THE UNIVERSITY OF ALBERTA

THE MEASUREMENT OF DUAL INPUT DESCRIBING FUNCTIONS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF ELECTRICAL ENGINEERING

EDMONTON, ALBERTA

JUNE, 1968

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled THE MEASUREMENT OF DUAL INPUT DESCRIBING FUNCTIONS submitted by Harry Gustav Hermann in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The research described in this thesis deals with an experimental method for the measurement of the dual input describing function. Applications of the measured DIDF to actual nonlinear feedback systems will also be given.

The output of the nonlinearity subject to a dual input will contain components with frequencies identical to the input and additional harmonics. By convention the low frequency input will be called the fundamental signal and the high frequency signal the stabilizing signal since the high frequency signal is usually used to regulate the fundamental output of the nonlinearity.

Two types of stabilizing signals have been used, the sinusoidal signal at 100 HZ and the stochastic signal with band width of 55 HZ to 550 HZ. Each of these signals has been passed through nonlinearities which ideally are frequency independent and time invariant. The output was measured and converted to the DIDF and then graphed.

The fundamental output of the nonlinearity subjected to a dual input may be measured by summing the output of the nonlinearity with the fundamental input of proper sign. Thus if the sum of these signals is measured with an RMS meter a minimum may be obtained by varying the fundamental. This minimum corresponds to the cancellation of the fundamental output from the nonlinearity output. If a band pass filter is used around the fundamental frequency a sharp minimum may be attained and a DIDF within ± 5% of the correct value calculated.

ACKNOWLEDGEMENTS

The author wishes to express his appreciation for the patient encouragement received from Professor Y. J. Kingma, under whose supervision this project has been completed.

The cooperation received from the staff and the graduate students of the Department of Electrical Engineering is gratefully acknowledged.

The author is indebted to the National Research Council and the University of Alberta for the financial assistance provided.

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I INTRODUCTION

This thesis is primarily concerned with the measurement of the dual input describing functions for both single and multiple valued nonlinearities. The method of measurement is applicable to any nonlinearity but in the interest of simplicity only time invariant and frequency independent nonlinearities with three or fewer parameters in their input output configurations will be considered.

To obtain an appreciation of the importance of the DIDF it is instructive to outline the development of the describing function.

The describing function was introduced to approximate the action of a nonlinearity in some nonlinear system. Since nonlinear elements such as relays, relays with dead band, limiters, limiters with dead band, etc., for example, are sturdy, reliable and relatively inexpensive, they are widely used in control applications as inexpensive amplifiers. As nonlinear elements became incorporated in nonlinear systems a method had to be developed to predict the stability of the systems. Goldfarb, Tustin, Oppelt and Kochenburger independently developed the describing function to provide an approximate analysis for the nonlinear systems (Ref. 1).

The describing function is merely the coefficient of the first term of the Fourier series describing the output of the nonlinearity divided by the amplitude of the fundamental input. The

NOMENCLATURE

$G(S)$	$\equiv$	the transfer function of the low pass filter following the nonlinearity
K	$\equiv$	the gain constant of the low pass filter
$x(t)$	$\equiv$	the input to the nonlinearity
$y(t)$	$\equiv$	the output of the nonlinearity
$P(A,B)$	$\equiv$	the amplitude of the fundamental component of the Fourier series representing the output of the nonlinearity
$DIDF(A,B)$	$\equiv$	the dual input describing function
A	$\equiv$	the amplitude of the fundamental sinusoid input to the nonlinearity
B	$\equiv$	the amplitude of the stabilizing signal input to the nonlinearity
σ	$\equiv$	the standard deviation or RMS value of the stochastic input signal to the nonlinearity
${}_1F_1$	$\equiv$	the confluent hypergeometric series
$E(A/B)$	$\equiv$	complete elliptical integral of the second kind
$K(A/B)$	$\equiv$	complete elliptical integral of the first kind
$\frac{\partial}{\partial A}$	$\equiv$	the partial differential with respect to A
Keq	$\equiv$	the ordinary describing function
C, G, F	$\equiv$	parameters used to describe the various nonlinearities

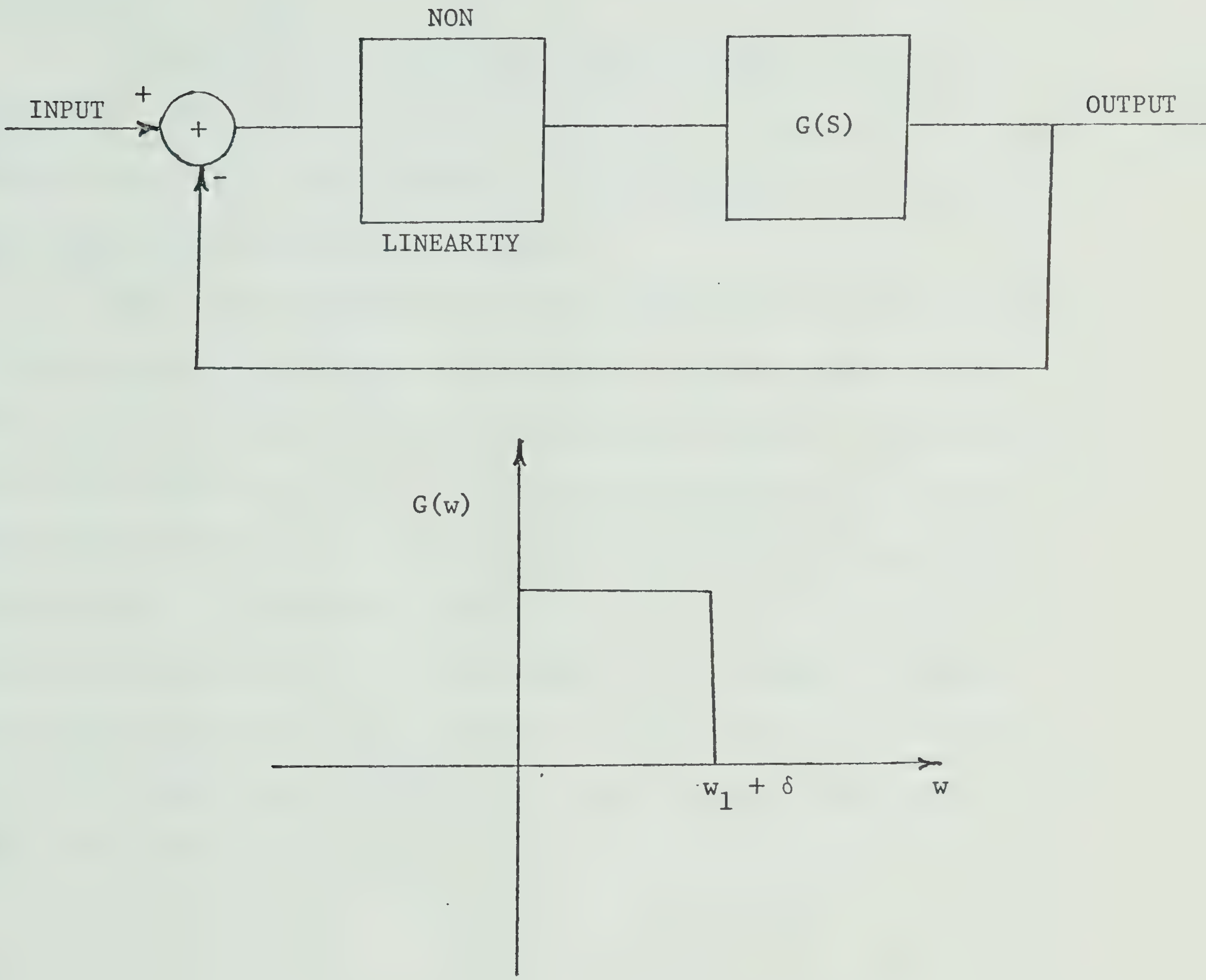
describing function is exact only in a system (Fig. 1.1) where an ideal low pass filter follows the nonlinearity and thus eliminates all fourier components other than the fundamental. Ideally then the describing function is the transfer function of the nonlinearity. In fact, the describing function has a limited accuracy.

Since some nonlinear systems exist which cannot be easily controlled by changing the gain without having undesirable side effects, it may be desirable to alter the fundamental output by introducing another stabilizing signal. Then again, there are some systems which already have a secondary signal such as an harmonic signal as in a practical feedback system or noise which may be introduced at the input. Thus having two inputs a dual input describing function can be defined as the ratio of the fundamental output of the nonlinearity to the fundamental input.

A dual input describing function can not only be used for two inputs but it can be extended to the ordinary describing function since in ordinary nonlinear feedback systems there are two or more inputs which may be significant depending on the low pass filter used.

The DIDF concept has been used by various authors, among them West, Douce and Livesley (Ref. 2) and later Oldenburger and Liu (Ref. 3) who introduced an equivalent gain concept very similar to the DIDF. It was defined as "the limit of the ratio of the average value of the output to the average value of the input as the average value of the input went to zero." This definition was elaborated on by Boyer (Ref. 4) who introduced a pseudo describing function which is identical with the dual input describing function described by Sridhar (Ref. 5). It was called a

FIGURE 1.1 The Exact Describing Function



w_2 is the radian frequency of the second harmonic of the output of the nonlinearity

w_1 is the radian frequency of the fundamental output of the nonlinearity

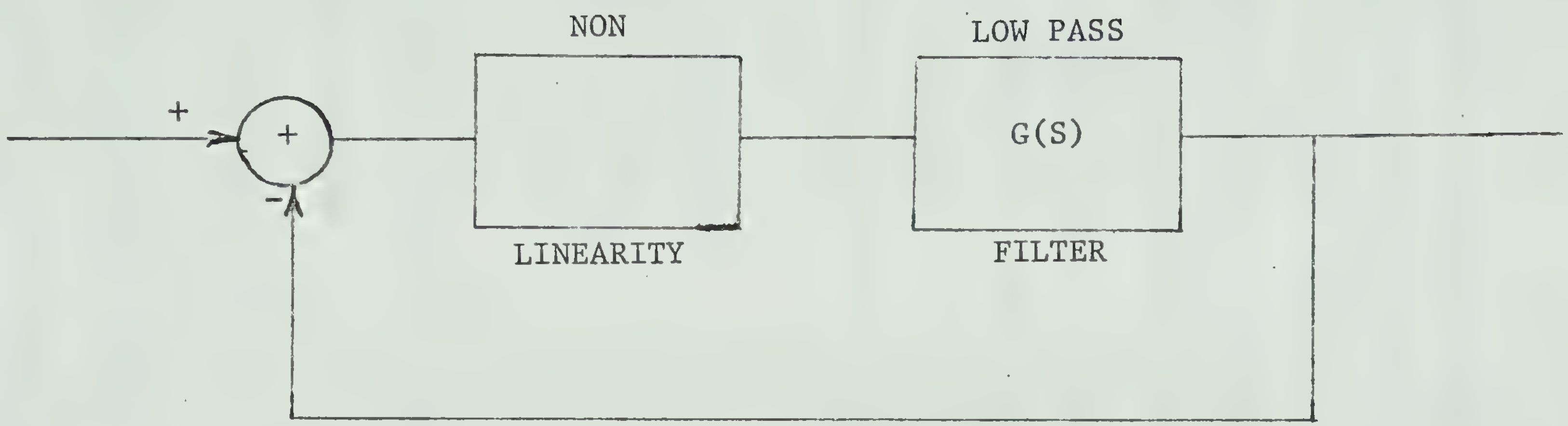
δ is an increment in radian frequency less than $w_2 - w_1$

pseudo describing function because the analytical method used in obtaining the function was an approximation to the exact function obtained by Sridhar. As it turned out the approximation was very close (identical for practical purposes).

Gibson (Ref. 6) describes a new DIDF which is identical by definition to the originally defined DIDF but he extends its use to forced systems disturbed by inputs within the operating range of the systems. This DIDF is used throughout this thesis.

This thesis will use a convenient experimental method for the determination of the DIDF of the relay, relay with dead band, limiter, limiter with dead band, and relay with hysteresis for both sinusoidal and stochastic stabilizing signals. In addition the stability of a simple nonlinear system (Fig. 1.2) containing the relay with hysteresis will be investigated. The nonlinear system will be forced with the two types of signals previously mentioned--the sinusoid and the stochastic signal. Analytical expressions for the DIDF for both the sinusoid and stochastic stabilizing signal exist for the relay, relay with dead band, limiter, and limiter with dead band (Ref. 5).

FIGURE 1.2 Simple Nonlinear System Containing the Relay with Hysteresis



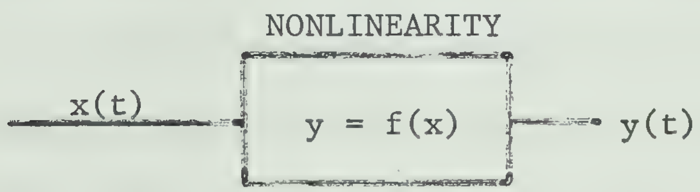
$$G(S) = \frac{K}{S(S + 10)^2}$$

II EXPERIMENTAL METHOD

a) Representation of the Nonlinearity

The nonlinearities, other than the relay with hysteresis, considered here are time invariant and frequency independent. The relay with hysteresis has a slight frequency dependence and the results given are not intended to simulate the ideal relay exactly.

In general the output of the nonlinearity can be analyzed and shown to have a fundamental output with the same frequency as the input and an infinite number of harmonics with decreasing amplitude. Thus the nonlinear device can be described as an input output device as follows:



$x(t) \equiv$ INPUT
$y(t) \equiv$ OUTPUT

The output y(t) of the nonlinearity may be separated arbitrarily into two components.

$$Y(t) = P(t) + H(t) \tag{2.1}$$

P(t) is the fundamental component of the output and has the same frequency as x(t). H(t) are all higher frequency harmonic outputs of the nonlinearity.

For the nonlinearity considered here $P(t)$ will at the most be a function of five variables--the amplitudes of the inputs plus the three parameters of the nonlinearities. In all of the cases two of the parameters may fortuitously be eliminated by normalizing the input amplitude with respect to parameter. Thus the three remaining variables may be graphed by using recurrent plots on a two dimensional graph.

b) Dual Input Describing Functions for Sinusoidal Stabilizing Signals

The DIDF considered here is the conventional describing function adapted for a dual input as first described by J. E. Gibson and R. Sridhar (Ref. 6) and incorporated in "Nonlinear Automatic Control" (Ref. 1, page 420). The DIDF is necessary to obtain some analytical approximation to the actual behaviour of a nonlinearity with a dual input. A single sine wave of fundamental frequency identical to the fundamental input frequency is usually the most logical choice to represent the output of a nonlinearity since sine waves can easily be handled mathematically. If this single sine wave is chosen as best in a minimum RMS-error sense, then its amplitude is the first Fourier coefficient of the fundamental output of the nonlinearity element (Ref. 1, page 380). If a dual input exists to the nonlinearity then the DIDF may be defined as

$$\mathrm{DIDF}(A,B) = \frac{P(A,B)}{A} \tag{2.2}$$

which holds only for a single valued nonlinearity. The dual input to the nonlinearity is $x(t)$.

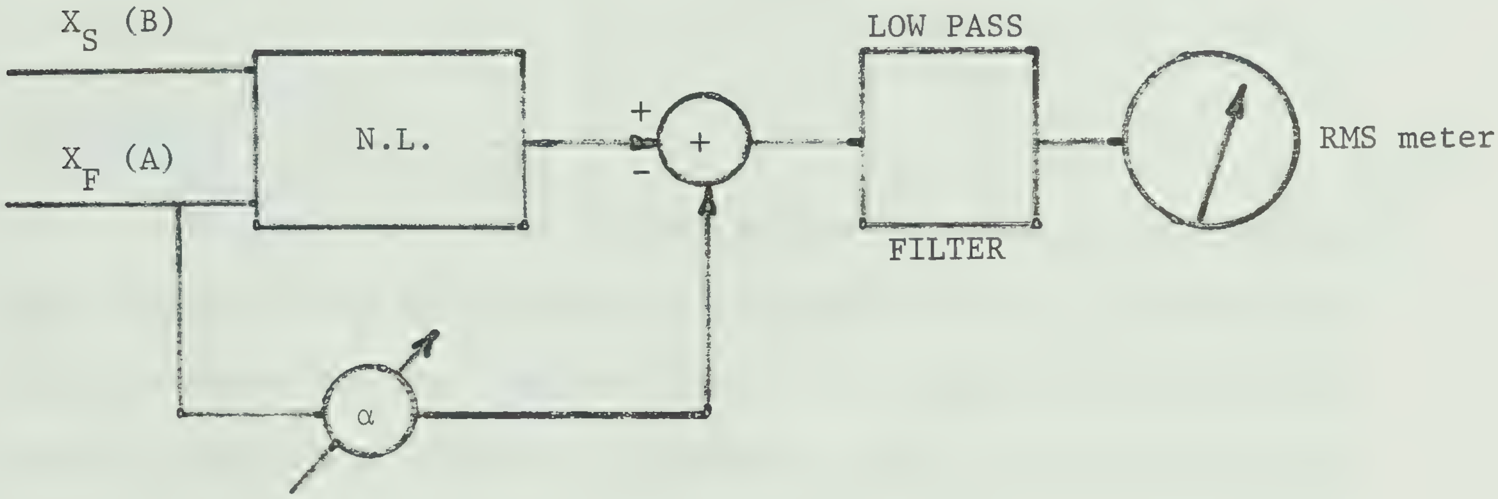
$$x(t) = X_F(A) + X_S(B) \tag{2.3}$$

$X_F = A \sin at$ $\qquad$ X_F = fundamental signal

$X_S = B \sin bt$ $\qquad$ X_S = stabilizing signal

P(A,B) is the fundamental output of the N. L.

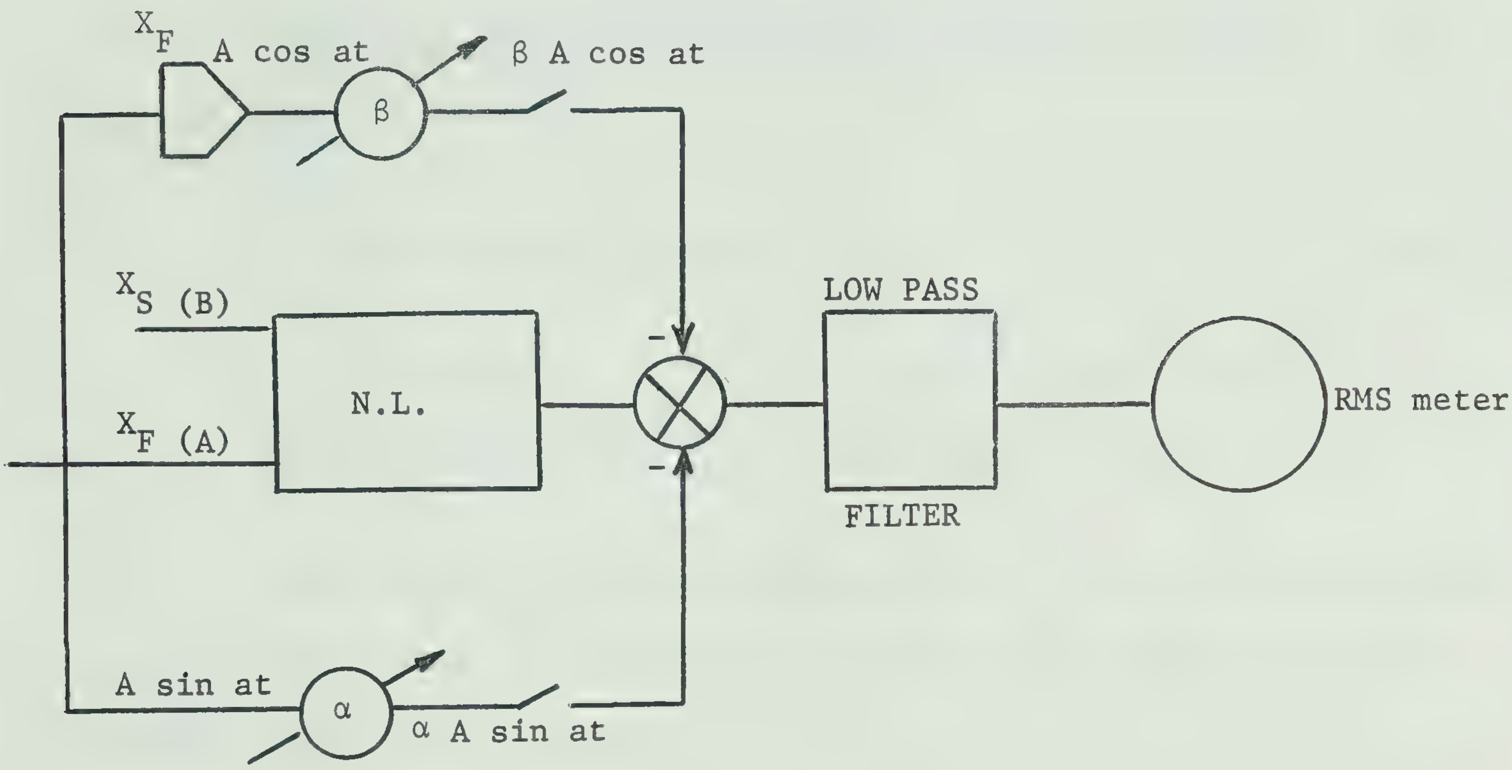
In order to measure the fundamental output we may use the following scheme for single valued nonlinearities.



The RMS meter reads a minimum when the fundamental output of the N. L. is cancelled from y(t). The amplitude of P(A,B) may subsequently be read from $\alpha \cdot X_F$. A better experimental method is to have the needle reach its original position. Thus the positional error in the meter is cancelled and $\alpha \cdot X_F$ may be divided by two.

Double valued nonlinearities require a signal 90° out of phase with X_F which may easily be obtained by passing X_F through an integrator on

the analogue computer.



The RMS meter reads a minimum when the real part of the fundamental output of the nonlinearity is cancelled from y(t). Similarly it reads a minimum when the imaginary part of the fundamental output is cancelled from y(T). The real and imaginary part may then be directly measured and the complex dual input describing function obtained. AS before a better experimental method is to have the meter return to its original position which corresponds to replacing the original fundamental output with a signal 180° out of phase. This allows a more accurate reading to be obtained for the DIDF.

$$\text{DIDF}\ (A,B) = \frac{R\ (P(A,B))}{A} + i\ \frac{\text{Im}\ (P(A,B))}{A} \qquad (2.4)$$

c) <u>Dual Input Describing Functions for Stochastic Inputs</u>

The stabilizing input to the nonlinearity may also be a stochastic input.

$$x(t) = X_F(t) + X_S(t) \tag{2.5}$$

X_F = A sin at $\qquad$ X_F = fundamental signal

X_S = stochastic signal $\qquad$ X_S = stabilizing input

The probability that the amplitude S of the stochastic signal X_S has a given value S_1 is given by the Gaussian Probability-Density Function P(x) (Ref. 1, page 51).

where $$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[\frac{-(x - \mu)^2}{2\,\sigma^2}\right]$$

$\mu \equiv$ average or expected value of the amplitude S

$\sigma \equiv$ standard deviation, a measure of scatter or dispersion

$\sigma^2 \equiv$ variance (second central moment)

$\mu = 0$ for the noise used in this thesis. Note that σ the standard deviation is also the RMS value of the noise.

Only noise with Gaussian amplitude distributions is used since it is easiest to handle analytically and since it occurs widely in nature. Besides, any non-Gaussian noise when passed through a band pass filter becomes Gaussian. Consequently Gaussian noise is assured at the output of the band pass filter.

The fundamental output of the nonlinearity subject to a dual input with Gaussian stabilizing signal may be treated in an identical manner to that for a sinusoidal stabilizing signal. The output of the nonlinearity will now be a function of the standard deviation σ of the noise and the fundamental input amplitude A.

$$\text{DIDF}\ (A,\sigma) = \frac{R(P(A,\sigma))}{A} + i\ \frac{Im(P(A,\sigma))}{A} \qquad (2.6)$$

III CONSIDERATION OF THE EXPERIMENTAL TECHNIQUE

a) General

To obtain an idea of the validity of the results given in Figures 3.1, 3.2, 3.6 to 3.13, and 3.15 to 3.18, it is instructive to examine the equipment and its synthesis.

The Pace 231R analogue computer containing linear passive components of ± 0.01% accuracy was used. The error due to physical components was not measurable. However, a true RMS reading meter was used for the measurement of the fundamental output and the fundamental input. This Brüel and Kjoer Type 2409 electronic voltmeter has an attenuator of accuracy better than 1% of full scale deflection. At the 10 HZ where all of the output measurements were made a measured accuracy of ± 5% was obtainable even for a stochastic stabilizing signal. The output fluctuated especially at low outputs when a stochastic stabilizing signal was used. Since analytical results exist for the single valued nonlinearities considered a quick comparison shows that a maximum error of ± 5% is easily attained. The close correspondence between analytical results and measured results is in itself a check on the ideality of the nonlinearity simulations which are considered shortly.

The errors introduced by the function generating equipment are minimal, however they will be mentioned here for the sake of completeness. A Hewlett Packard 200CD Wide Range Oscillator was used to generate the 100 HZ signal. It had a distortion less than 0.5%. The noise was

generated by a G. R. random noise generator TYPE 1390-B and was passed through a band pass filter (Fig. 3.3) (55-550 HZ) which cut off 60 db per decade on the low frequency side. This sharp cutoff was necessary to prevent excessive fluctuations in the output of the nonlinearity.

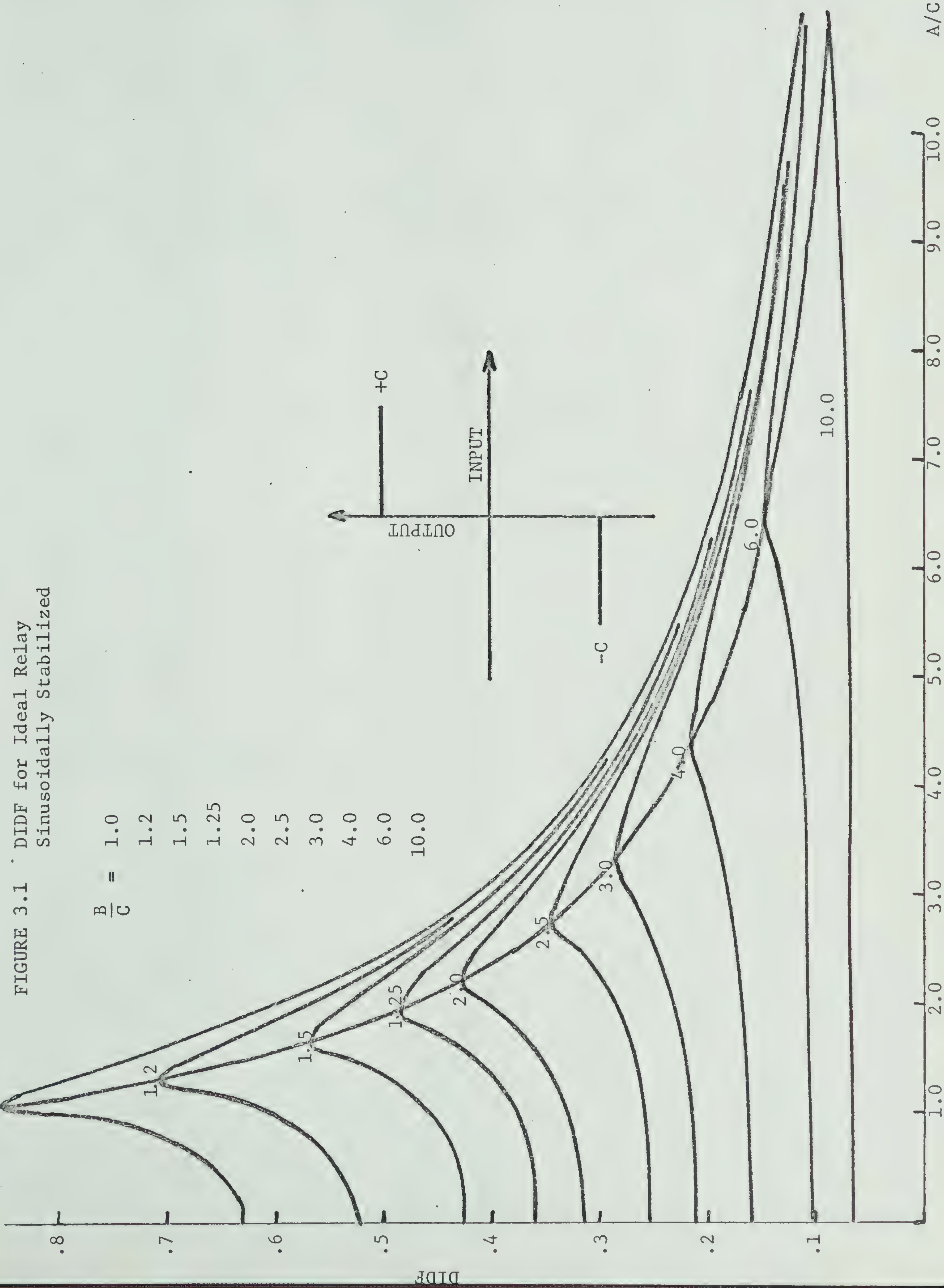
The measurement technique is shown in Fig. 3.4. All measurements were done with one meter which was switched back and forth in the arrangement shown. Pots P_1 and P_2 were varied to obtain the desired input. Switch 4 was used to introduce an imaginary component to the output and to the measuring device. P_4 was used to obtain a minimum reading on the RMS meter.

b) <u>The Relay</u>

Simulation of the relay is given in Fig. 3.5. The feedback and series limiters sufficiently hardened the output so that there was a net change of less than 10 millivolts in the output for a 100 volt change in the input. The nonlinearity however was slightly frequency dependent. The relay had a finite rise time and settling time which varied as the input varied. In general the frequency dependence was not noticeable for inputs above 10 millivolts and frequency of 100 HZ or less.

The error was largest for low inputs, of course, but never exceeded the ± 5% range. The graphs of the DIDF vs. the amplitude input are self explanatory, however they do differ. The slope of the DIDF vs. input amplitude graph for a stochastic stabilizing signal is always negative while the slope of the DIDF vs. input amplitude graph for the sinusoid stabilizing signal has both positive and negative slopes. This phenomenon

FIGURE 3.1 DIDF for Ideal Relay Sinusoidally Stabilized



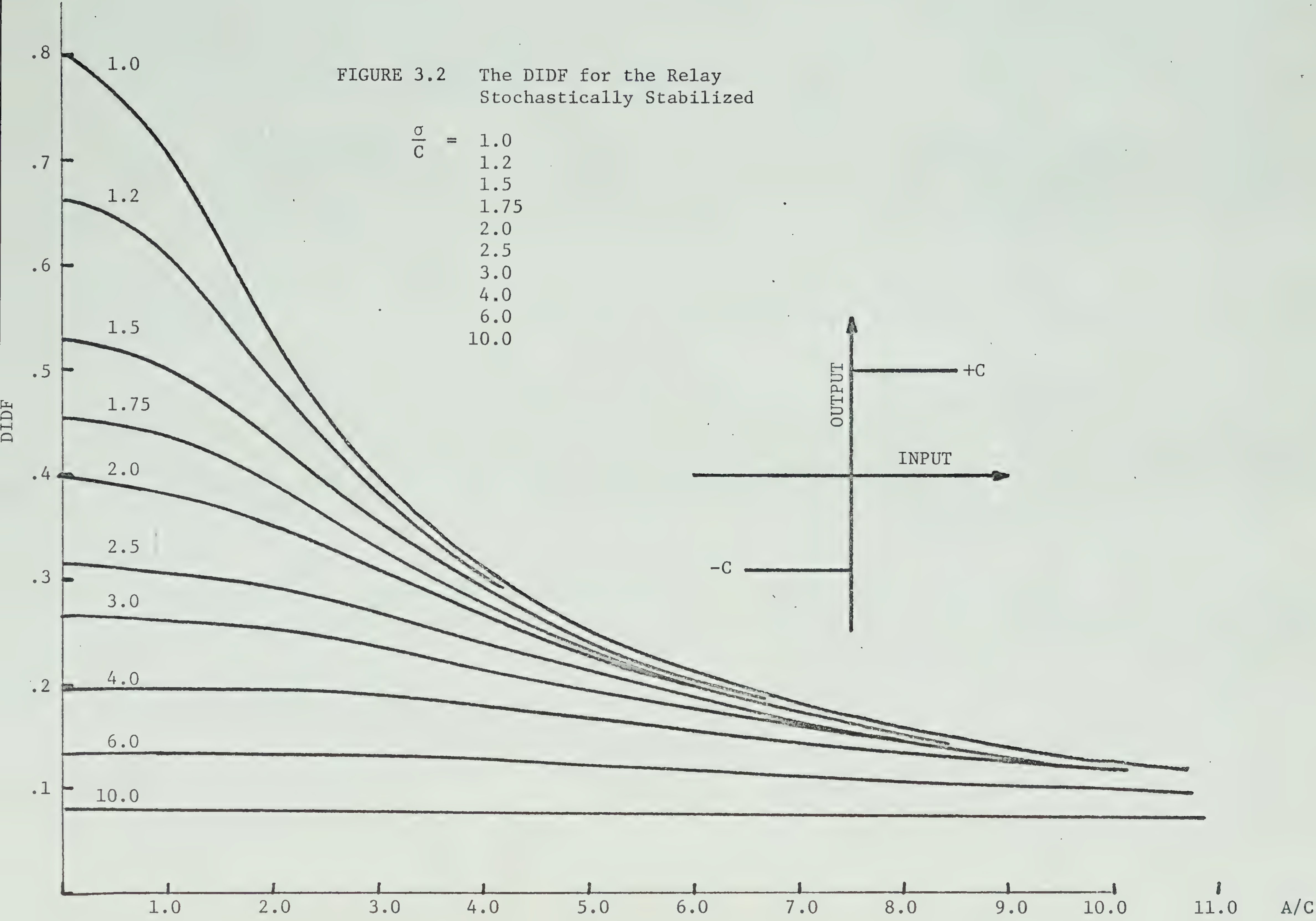
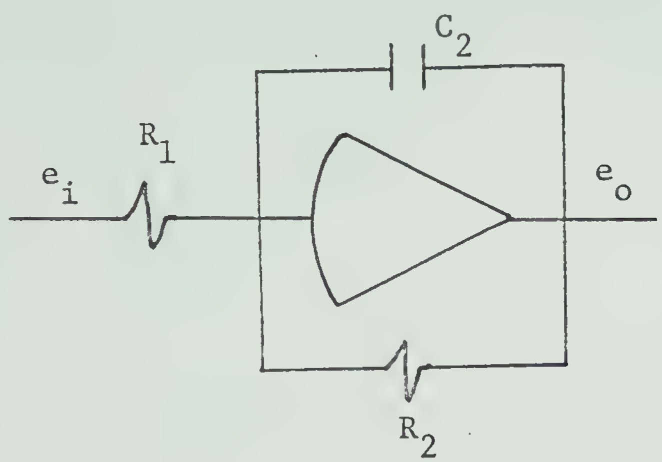


FIGURE 3.2 The DIDF for the Relay Stochastically Stabilized

FIGURE 3.3 The Shaping Filter for the Noise
55-550 HZ

Low Pass (2 units)



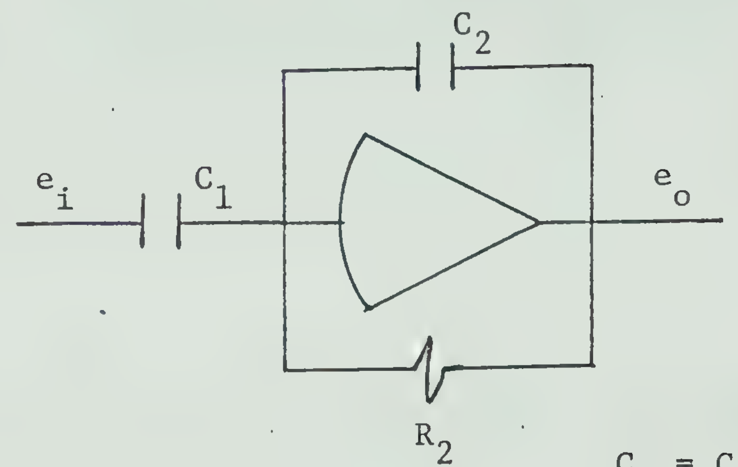
$$\frac{e_o}{e_i} = -\frac{R_2/R_1}{1 + S\ R_2 C_2}$$

R_1 = .65 MΩ

R_2 = .65 MΩ

C_2 = .001 μf

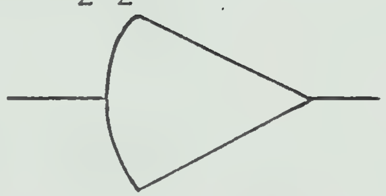
High Pass (3 units)



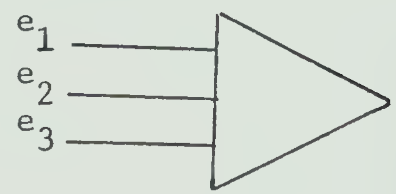
$C_1 = C_2 = 0.1$ μf

$R_2 = 0.50$

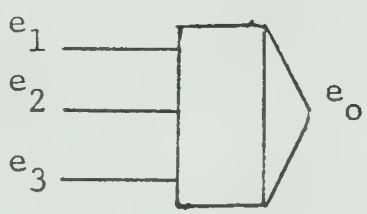
$$\frac{e_o}{e_i} = -\frac{SC_2R_2}{S\ C_2R_2 + 1}$$



High Gain Amplifier

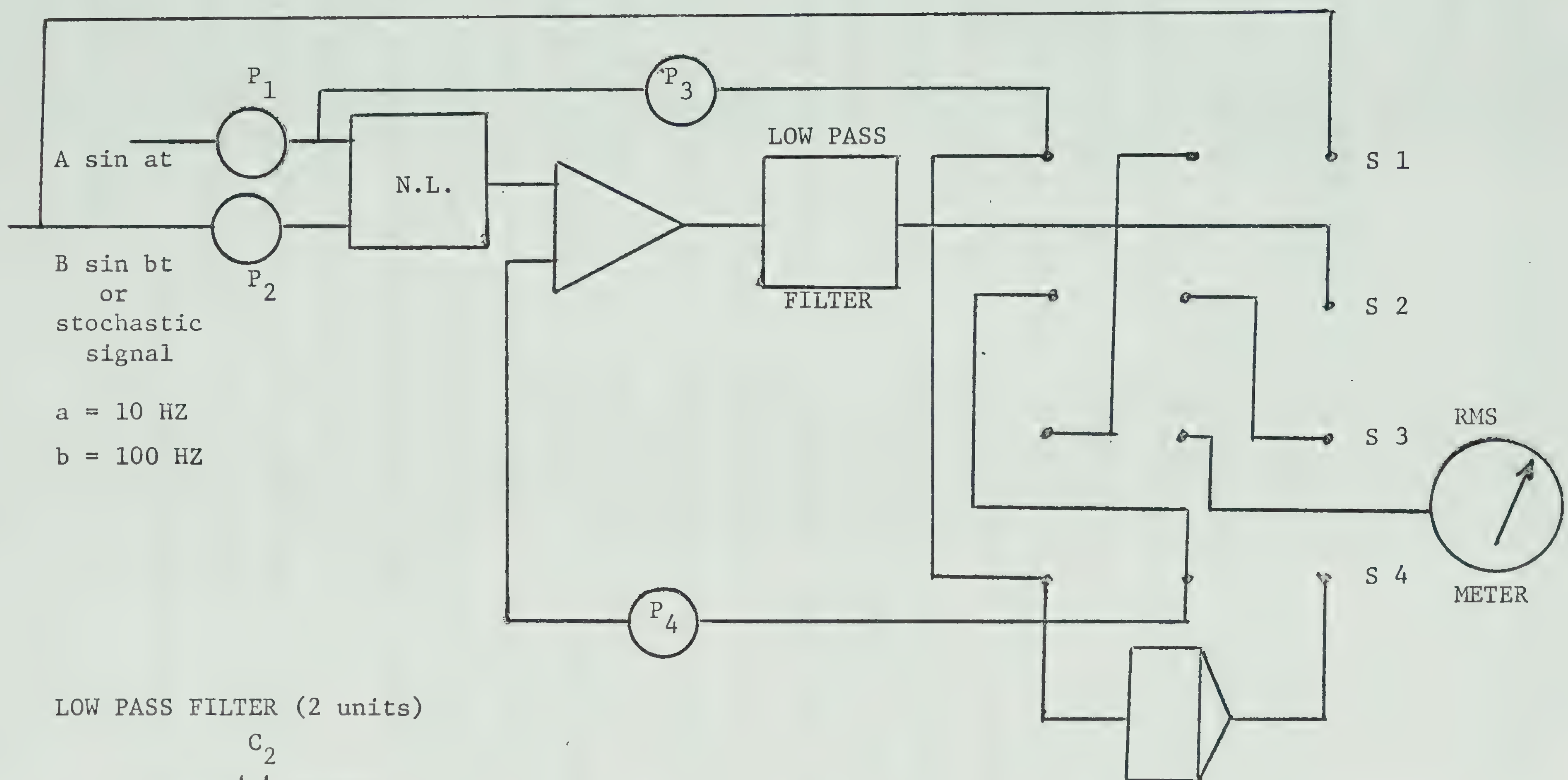


Summer $e_o = -(e_1 + e_2 + e_3)$

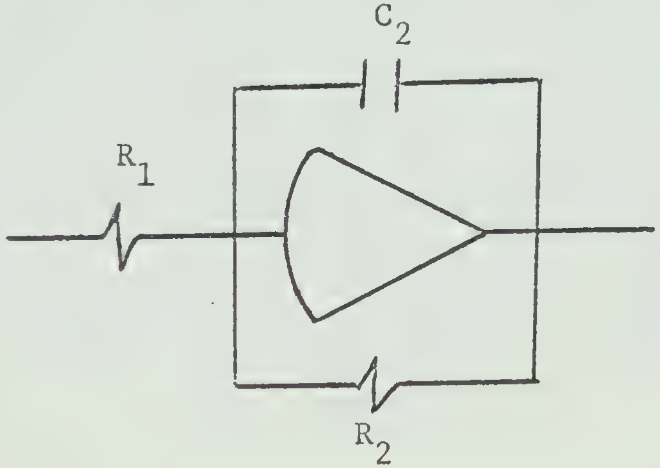


Integrator $e_o = -\int_o^t (e_1 + e_2 + e_3)dt$

FIGURE 3.4 The Analogue Measuring Setup



LOW PASS FILTER (2 units)



$$\frac{e_o}{e_i} = \frac{-R_2/R_1}{1 + S\,R_2C_2}$$

$R_2 = R_1 = .150\ M\Omega$

$C_2 = .1\ \mu f$

$f_1 \approx 10.5\ HZ$

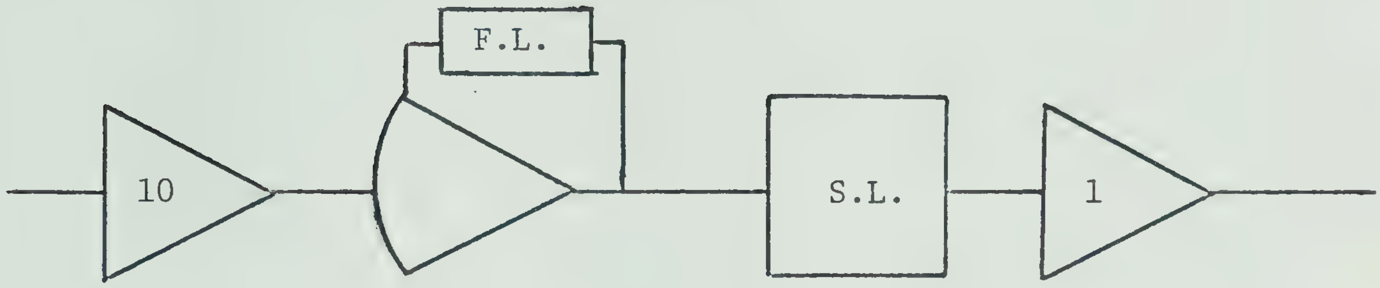
and its application will be investigated further in a later part of this thesis.

c) The Relay with Dead Band

The relay with dead band simulation is shown in Fig. 3.5. The paths of (+) and (-) incoming signals greater than some predetermined minimum, set by the first set of reference pots, is shown. Two amplifiers are ganged because the series limiter must end in an impedance of 1 MΩ which can only be realized with a gain of 1. The gain of 10 is necessary to drive the diodes into saturation completely. The second set of pots can be used individually to compensate for voltage drops in the diodes or maladjustments of the feedback and series limiters. Since this adjustment is rather time consuming it is convenient to have these pots there.

Upon testing the ideal set up against calculated values it was found that the relay did not respond exactly at an amplitude input equal to the dead band. At this point the relay output is very sensitive to changes in amplitudes of the input and it was found convenient to end the dead band a little earlier so that the finite rise time of the relay could be partially offset. The relay was again checked and found to correspond quite closely to the calculated values. This adjustment is most conveniently done experimentally by triggering the relays at a lower value of the dead band. This correction brings the output of the relay for lower inputs closer to calculated values while it worsens the output for higher inputs. However, the error is now more evenly distributed over the range of the output of the relay.

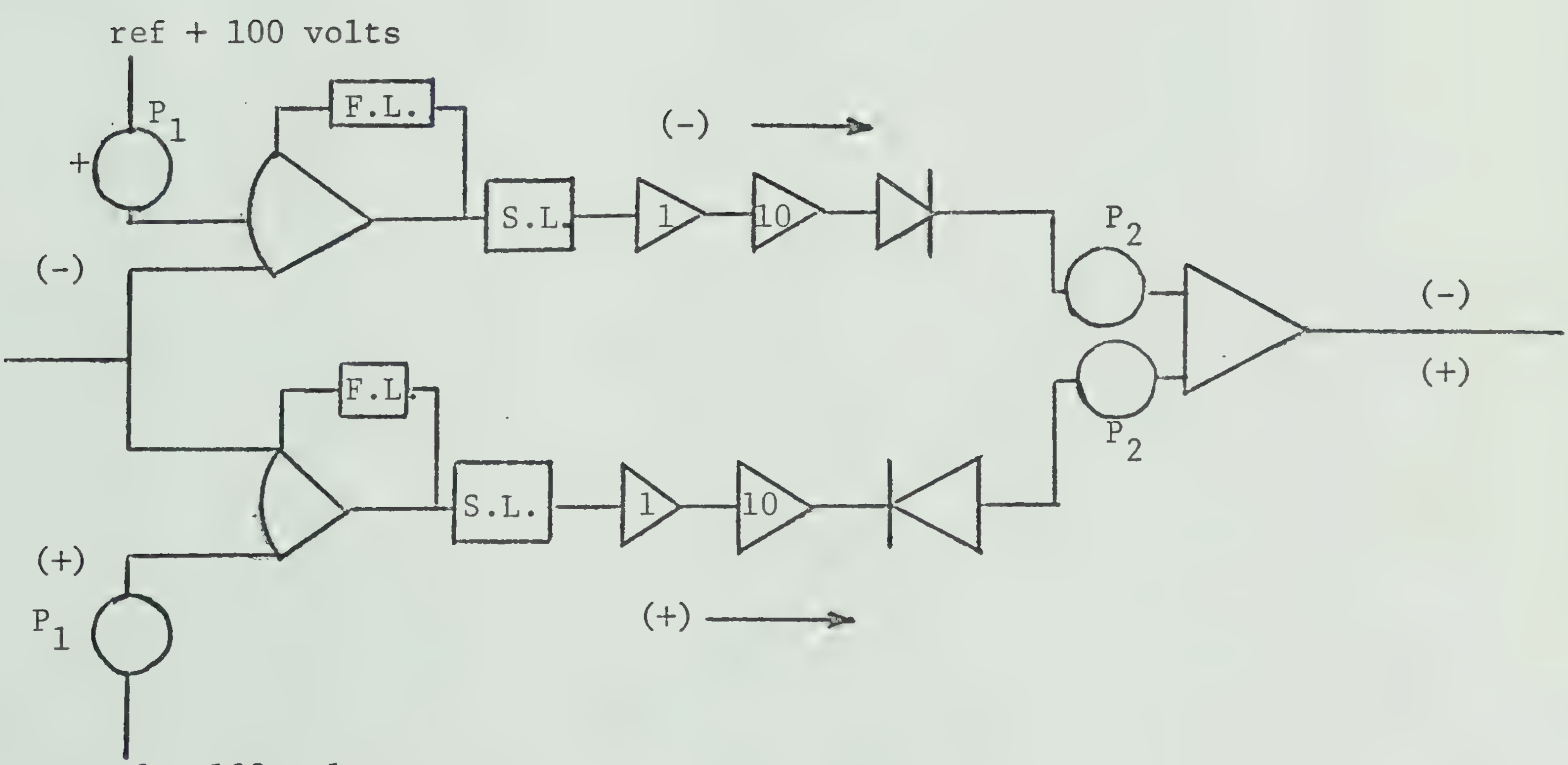
FIGURE 3.5 Analogue Simulation of the Relay



F.L. ≡ feedback limiter

S.L. ≡ series limiter

Analogue Simulation of the Relay with Deadband



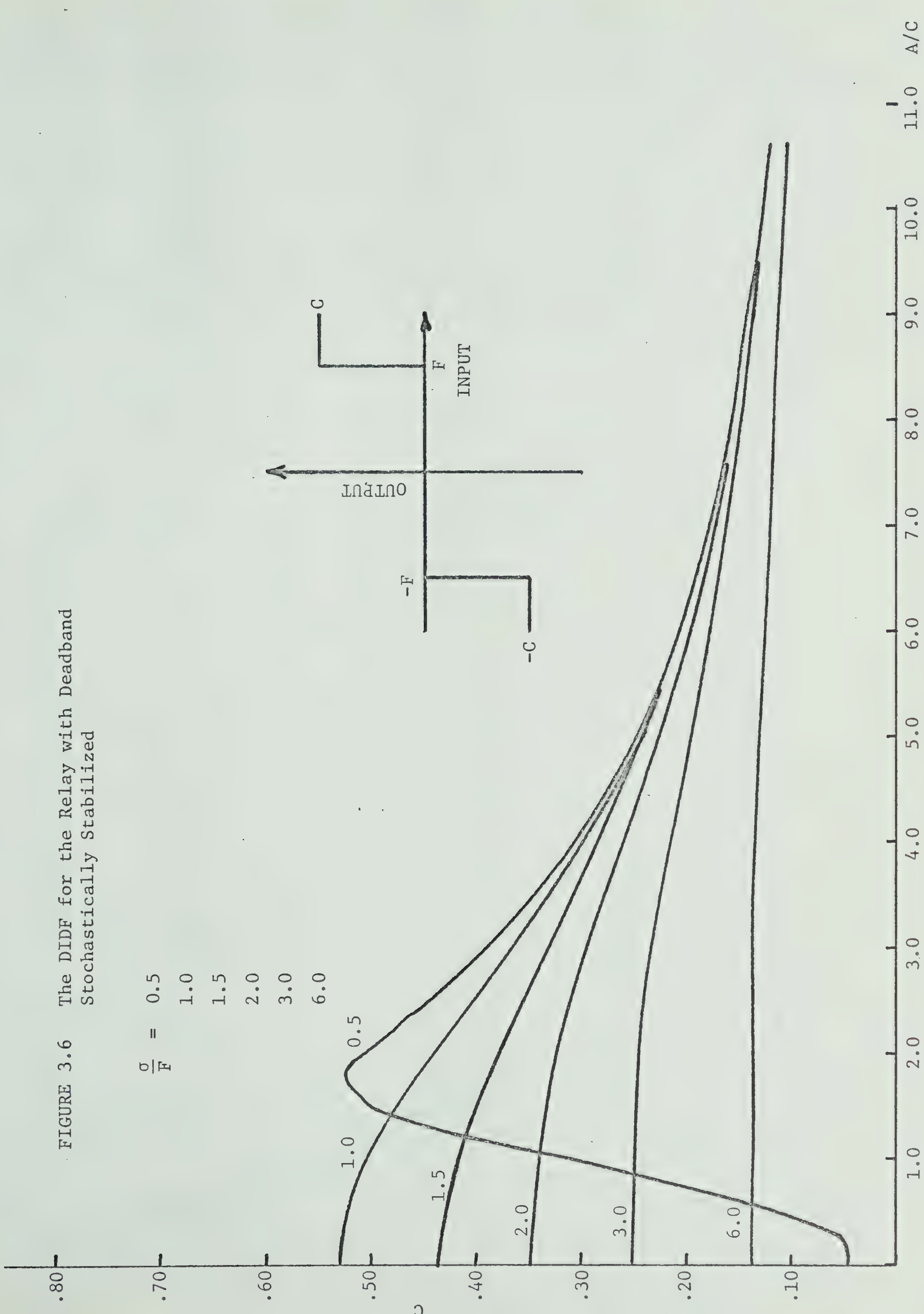
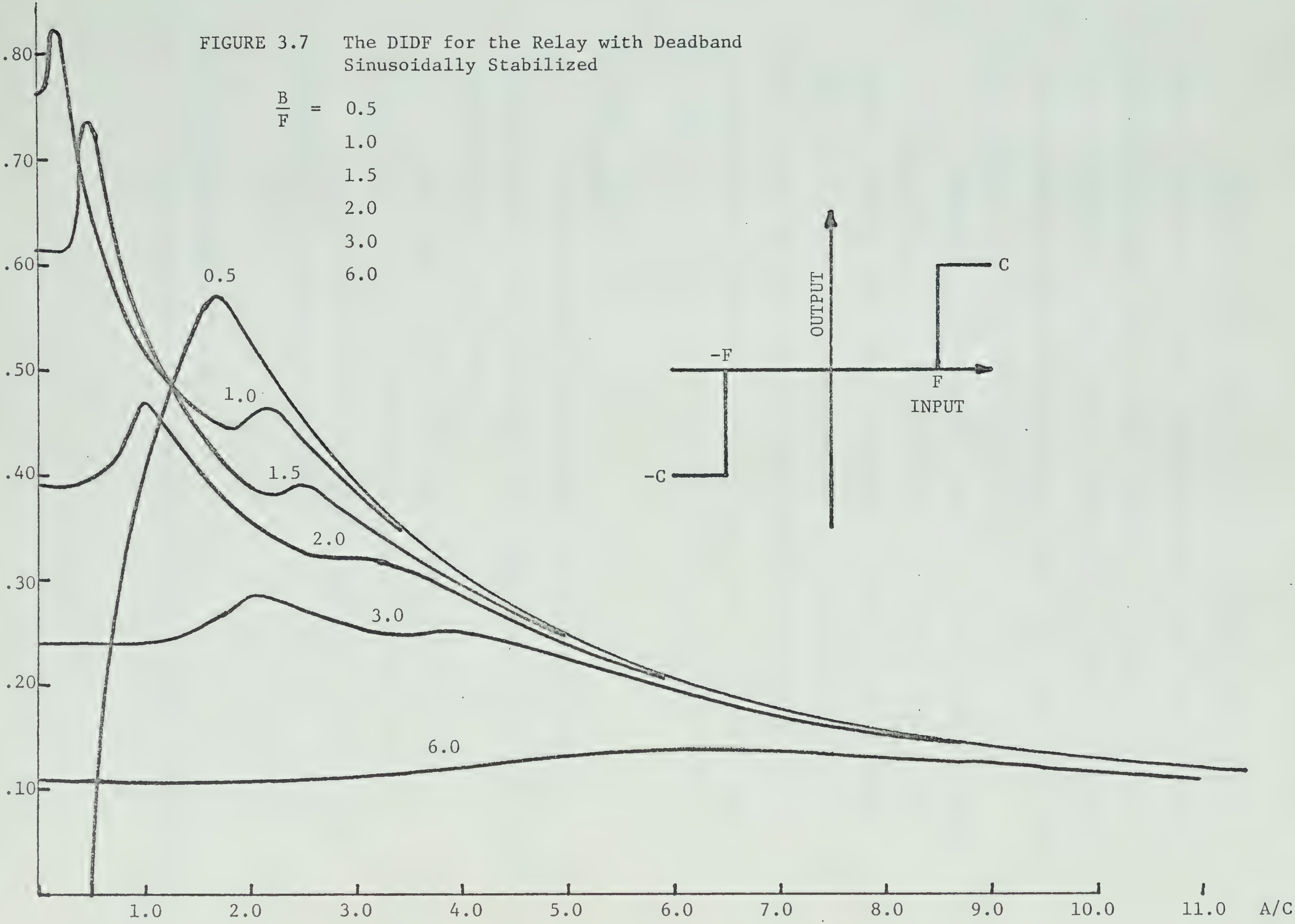


FIGURE 3.6 The DIDF for the Relay with Deadband Stochastically Stabilized

FIGURE 3.7 The DIDF for the Relay with Deadband Sinusoidally Stabilized



A quick check with the scope shows a rise time which varies inversely with the amplitude of the input between 5 and 1 ms. A weighted mean would be around 2 ms. In order to correct the output for the 2 ms delay in triggering one would have to change the dead band of the relay as one changed the amplitude of the input. Since this procedure is impractical the experimental approach is justified.

The DIDF for the relay was plotted for both a sinusoidal signal and stochastic signal. The sinusoidal plot is in general more accurate than the stochastic plot since the stochastic signal had higher frequencies which, as noted, the relay cannot respond to. Additionally, of course, the meter readings for the stochastic signal fluctuated slightly about an average because of the variation in the power output of the noise generator. On the whole the readings were within ± 5% of the actual calculated values.

d) The Limiter

The ideal limiter was reproduced on the analogue computer as shown in Fig. 3.14. The set up is quite straightforward and the errors are minimal because of its simplicity.

The results for the limiter were extremely accurate and the largest error was due to the measurement technique and the meter used.

The plots for the sinusoidal and stochastic inputs were quite straightforward and need no comment here.

e) The Limiter with Dead Band

The simulation of the relay with dead band is shown in Fig. 3.14.

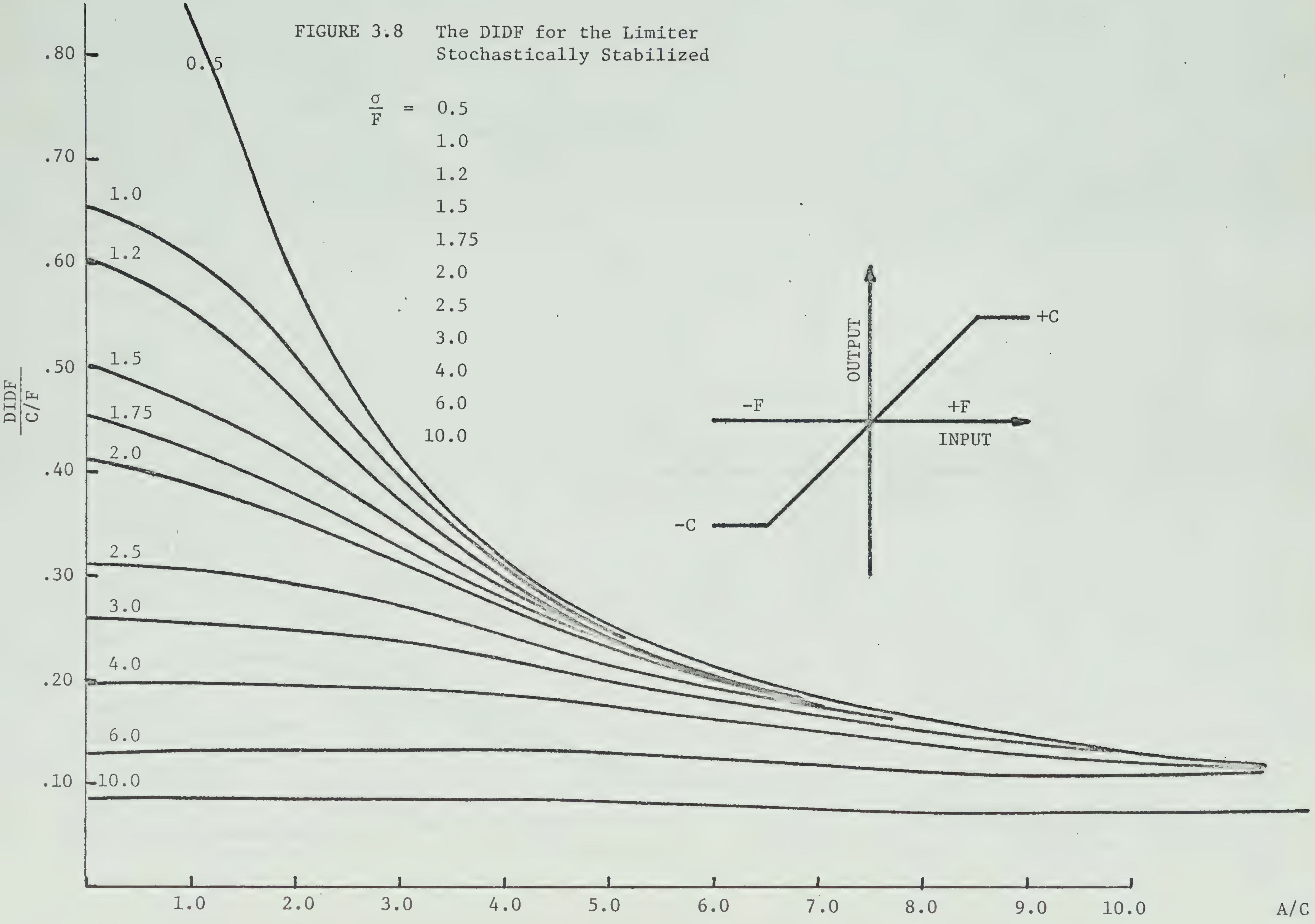


FIGURE 3.8 The DIDF for the Limiter Stochastically Stabilized

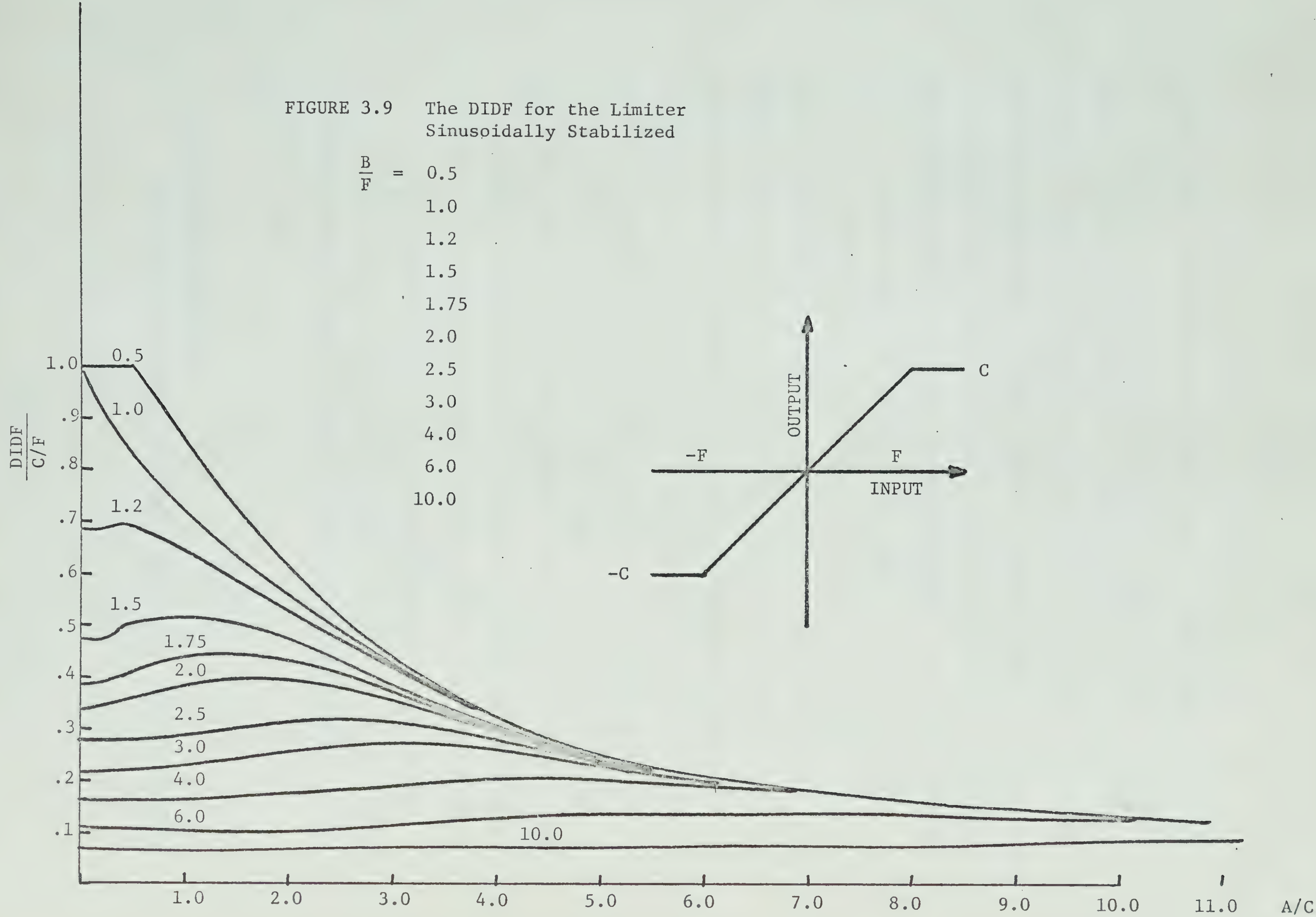


FIGURE 3.9 The DIDF for the Limiter Sinusoidally Stabilized

The first set of pots set the dead band while the second gave the limiter the correct slope and the third attenuated the signal to the proper limiting value. Again two amplifiers were used in series plus a gain of 10 on the amplifier with the feedback limiters to give the necessary gain to saturate the diodes even for small inputs. A quick check on the linear part of the limiter can be accomplished by bypassing the second pot and everything following, and feeding into the last amplifier. The output will be zero for operating conditions in the linear part of the limiter if the slope of the limiter is 1. The procedure is applied separately to each branch of the analogue simulation. This procedure shows there is a variable error in the output of ± 0.7 maximum volts when the limiter is operating linearly. This error occurs because the diodes are not completely hard and their output varies slightly as they are saturated more and more. Consequently for very small outputs of the nonlinearity there could be a high percentage error. The results of the measurement technique were checked with calculated values. There were no pronounced deviations although the errors tended to be largest for small values of output. On the whole the measured values were within ± 5% of the actual values.

The accuracy of the lower measurements was increased because the output tended to be more linear the larger the stabilizing signal used. Consequently readings could be taken at larger values and extrapolated to the smaller ones.

f) The Relay with Hysteresis

The analogue simulation is shown in Fig. 3.19. The four single

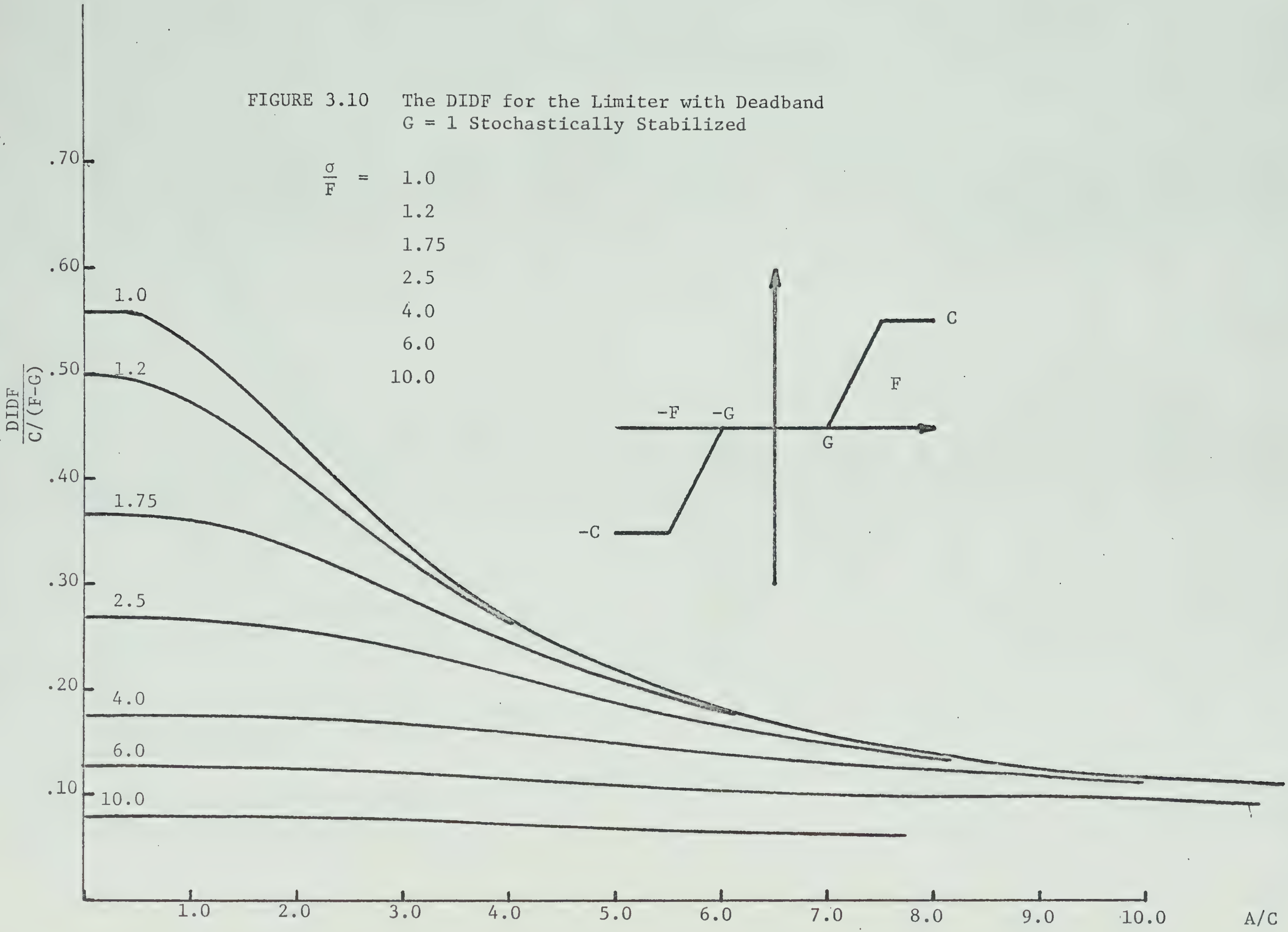


FIGURE 3.10 The DIDF for the Limiter with Deadband
G = 1 Stochastically Stabilized

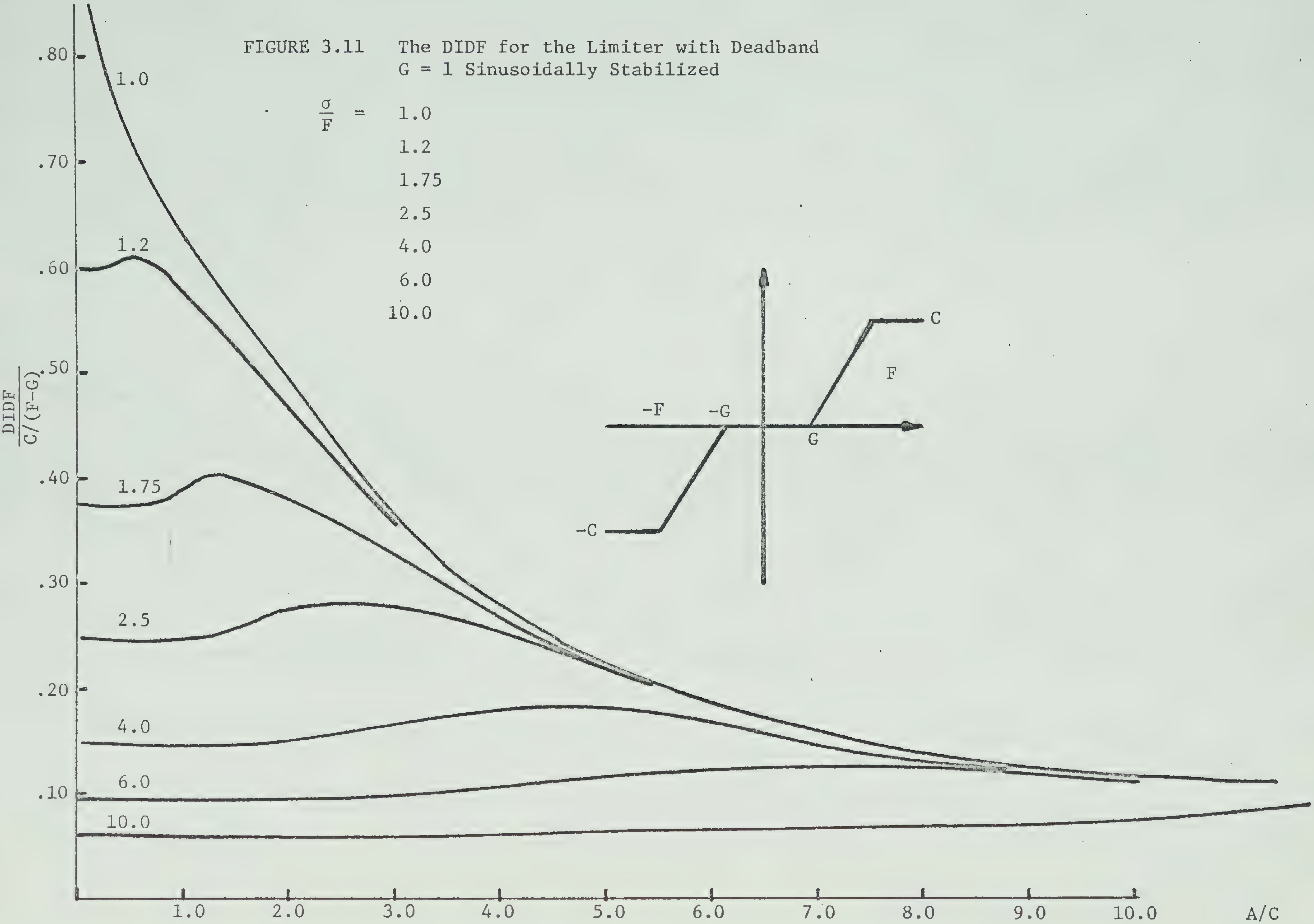


FIGURE 3.11 The DIDF for the Limiter with Deadband
G = 1 Sinusoidally Stabilized

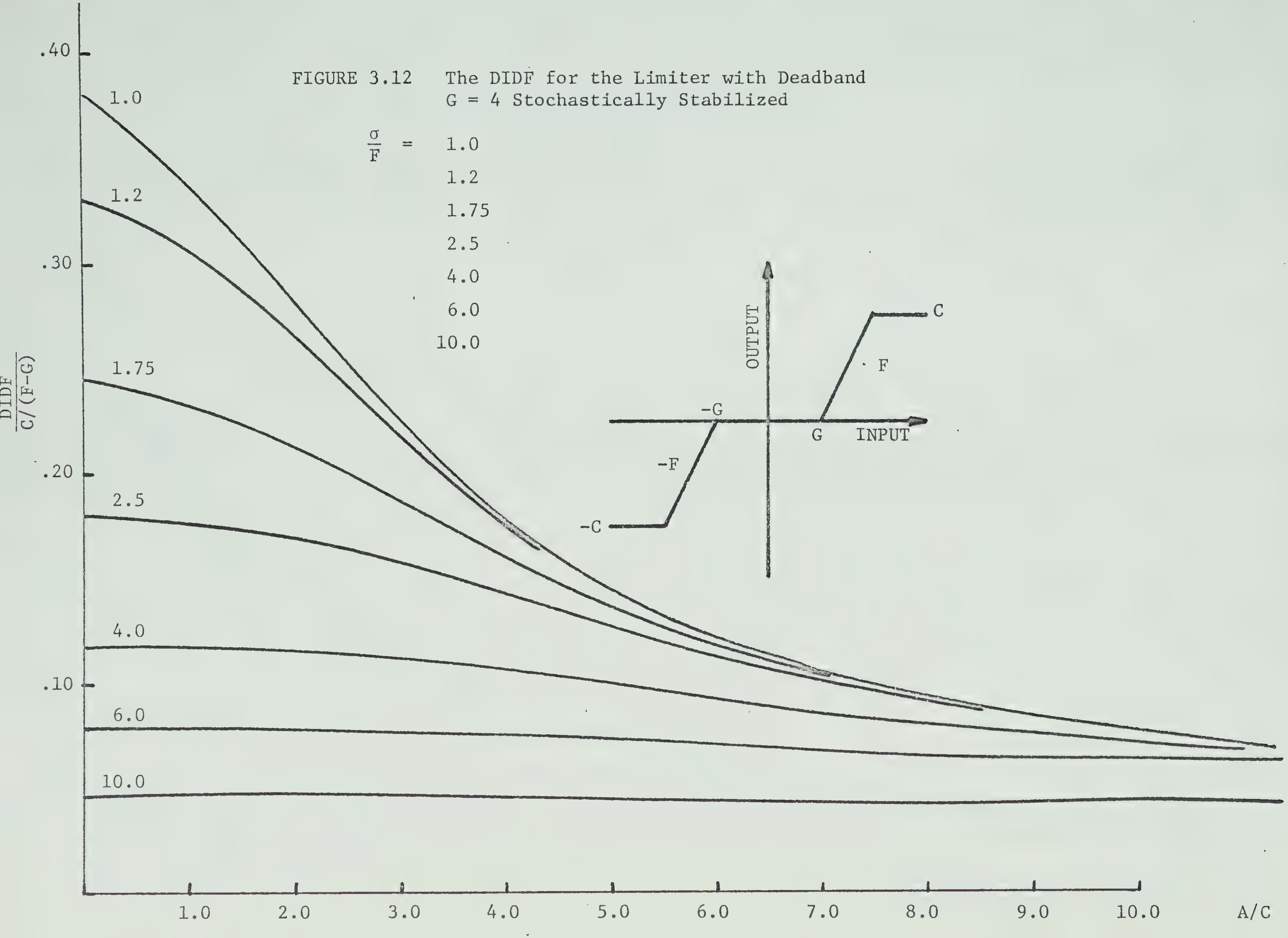


FIGURE 3.12 The DIDF for the Limiter with Deadband
G = 4 Stochastically Stabilized

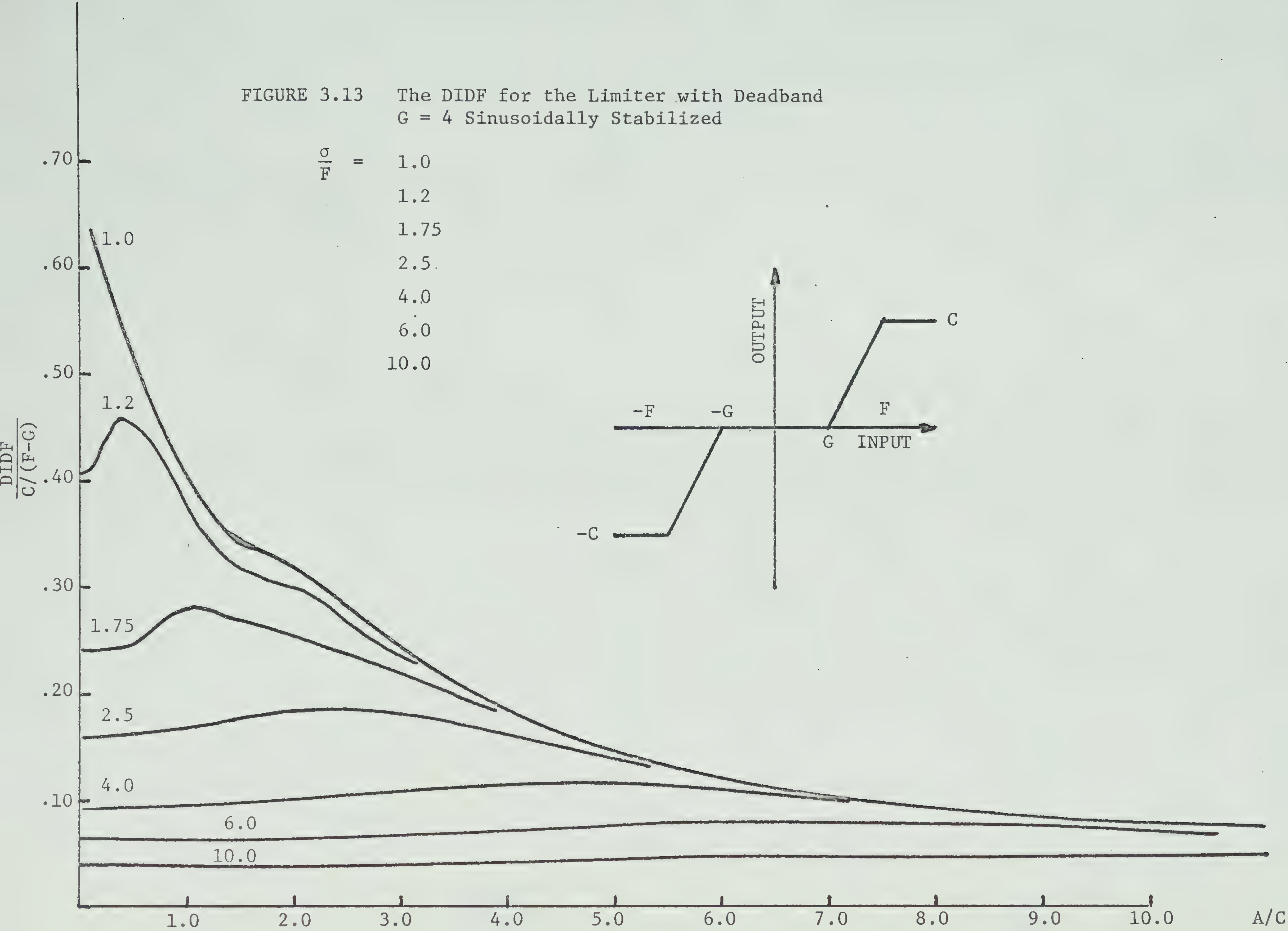
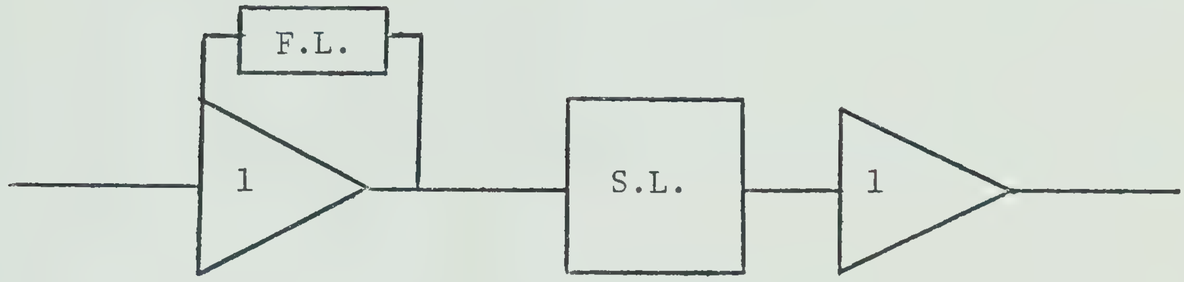


FIGURE 3.13 The DIDF for the Limiter with Deadband G = 4 Sinusoidally Stabilized

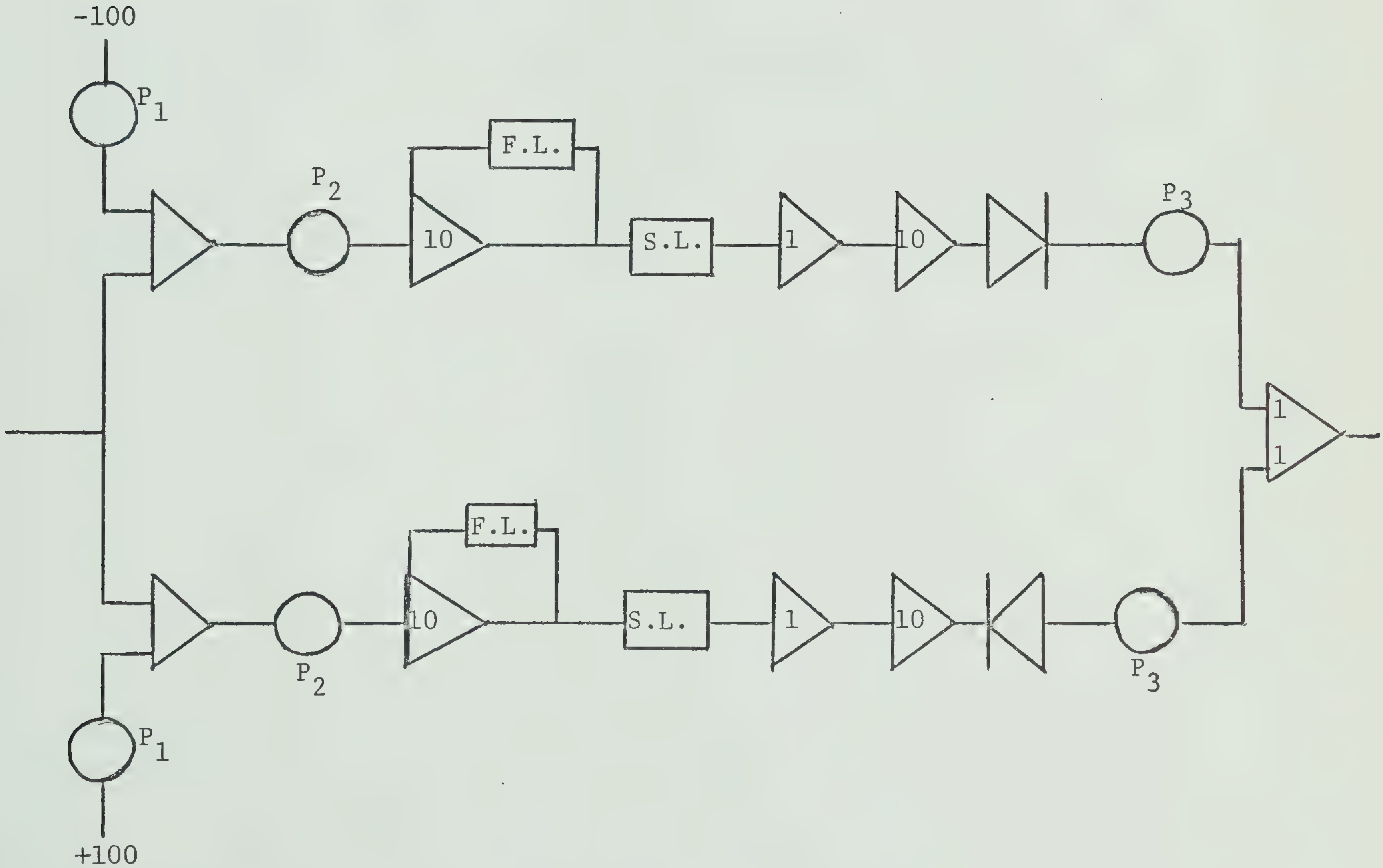
FIGURE 3.14 Analogue Representation of the Limiter

F.L. ≡ Feedback Limiter

S.L. ≡ Series Limiter



Analogue Representation of the
Limiter with Deadband



valued nonlinearities were easily set up. The relay with hysteresis however could not be made to function properly with an error below ± 5%. The delay involved in the finite rise times and finite settling times could not be adjusted to give a correct output at 10 HZ. Consequently the relay with hysteresis was set up ideally for a D.C. input and was treated as a unique nonlinearity at 10 HZ.

After all, the method for measurement described previously is applicable to all nonlinearities and a good test of its effectiveness is to work with a nonideal relay.

Using the previously described measuring technique it was found that the imaginary component of the output when there was a dual input was too small to measure. Consequently only graphs for the real part of the DIDF are shown. To check on the validity of the measurements the nonideal relay was placed in a feedback loop as in Fig. 1.2 and its stability examined experimentally and from the DIDF graphs previously obtained.

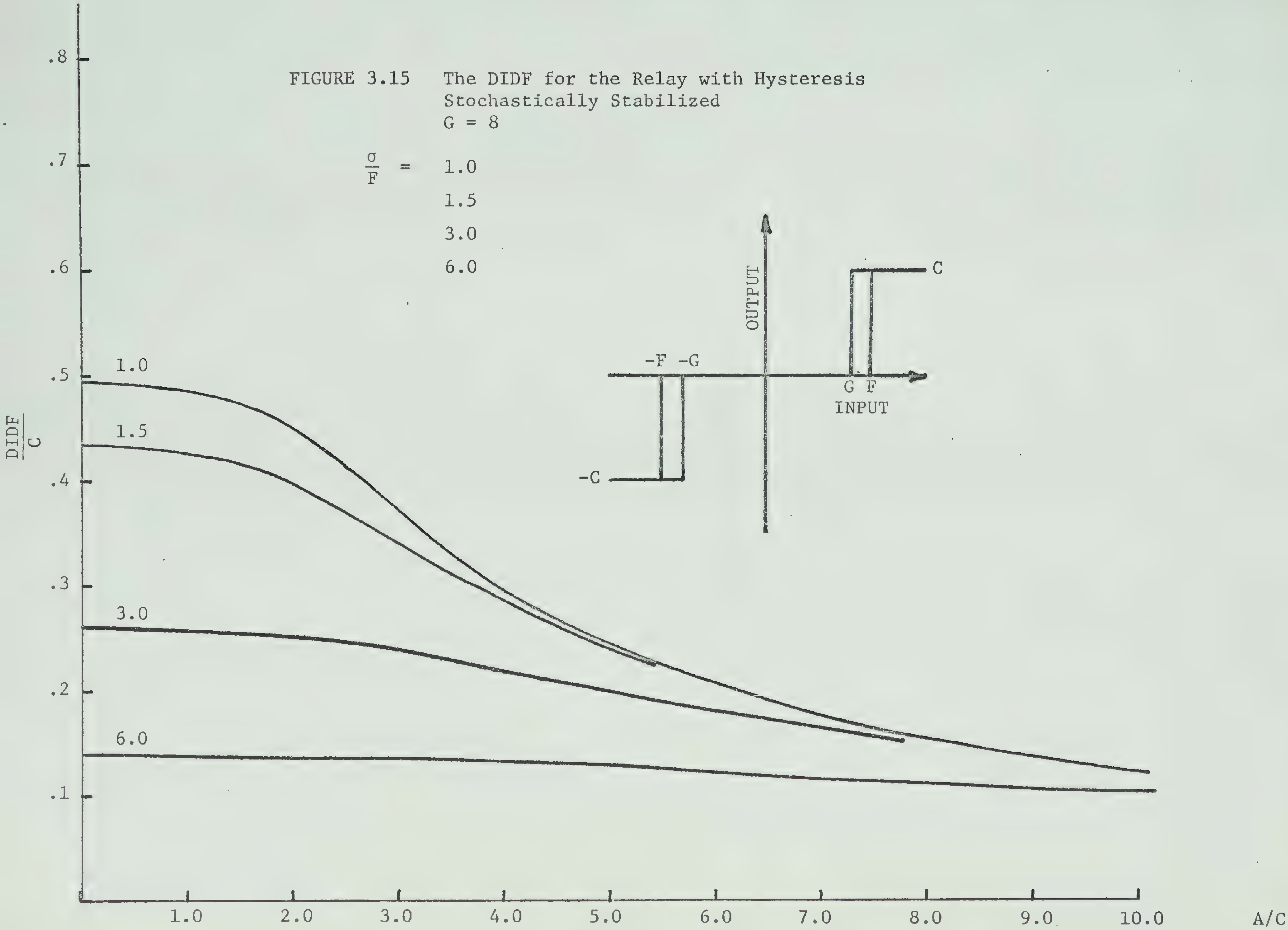


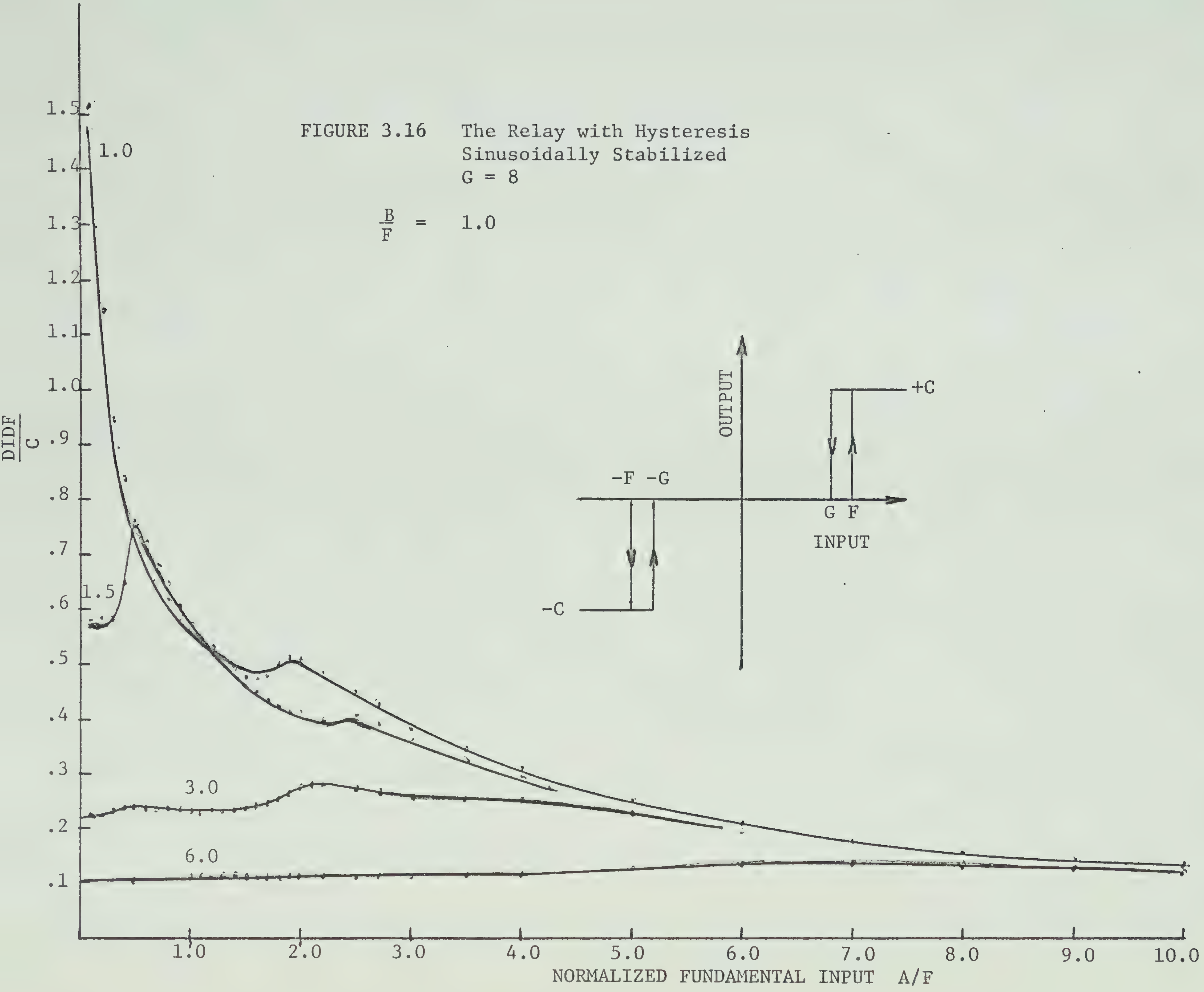
FIGURE 3.15 The DIDF for the Relay with Hysteresis Stochastically Stabilized

$G = 8$

$\frac{\sigma}{F}$ = 1.0, 1.5, 3.0, 6.0

FIGURE 3.16 The Relay with Hysteresis Sinusoidally Stabilized
G = 8

$\frac{B}{F}$ = 1.0



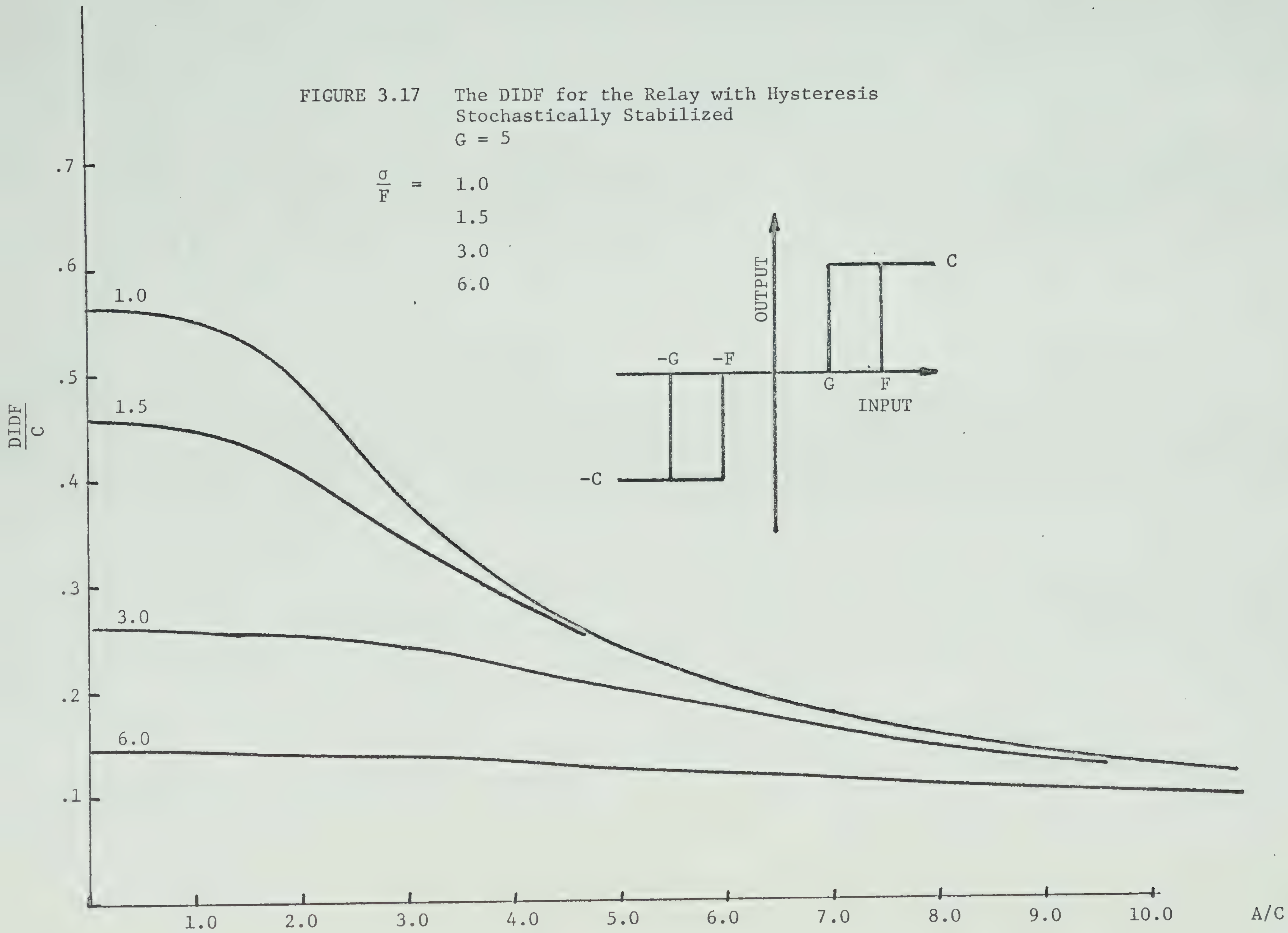


FIGURE 3.17 The DIDF for the Relay with Hysteresis Stochastically Stabilized

$G = 5$

$\frac{\sigma}{F} =$ 1.0, 1.5, 3.0, 6.0

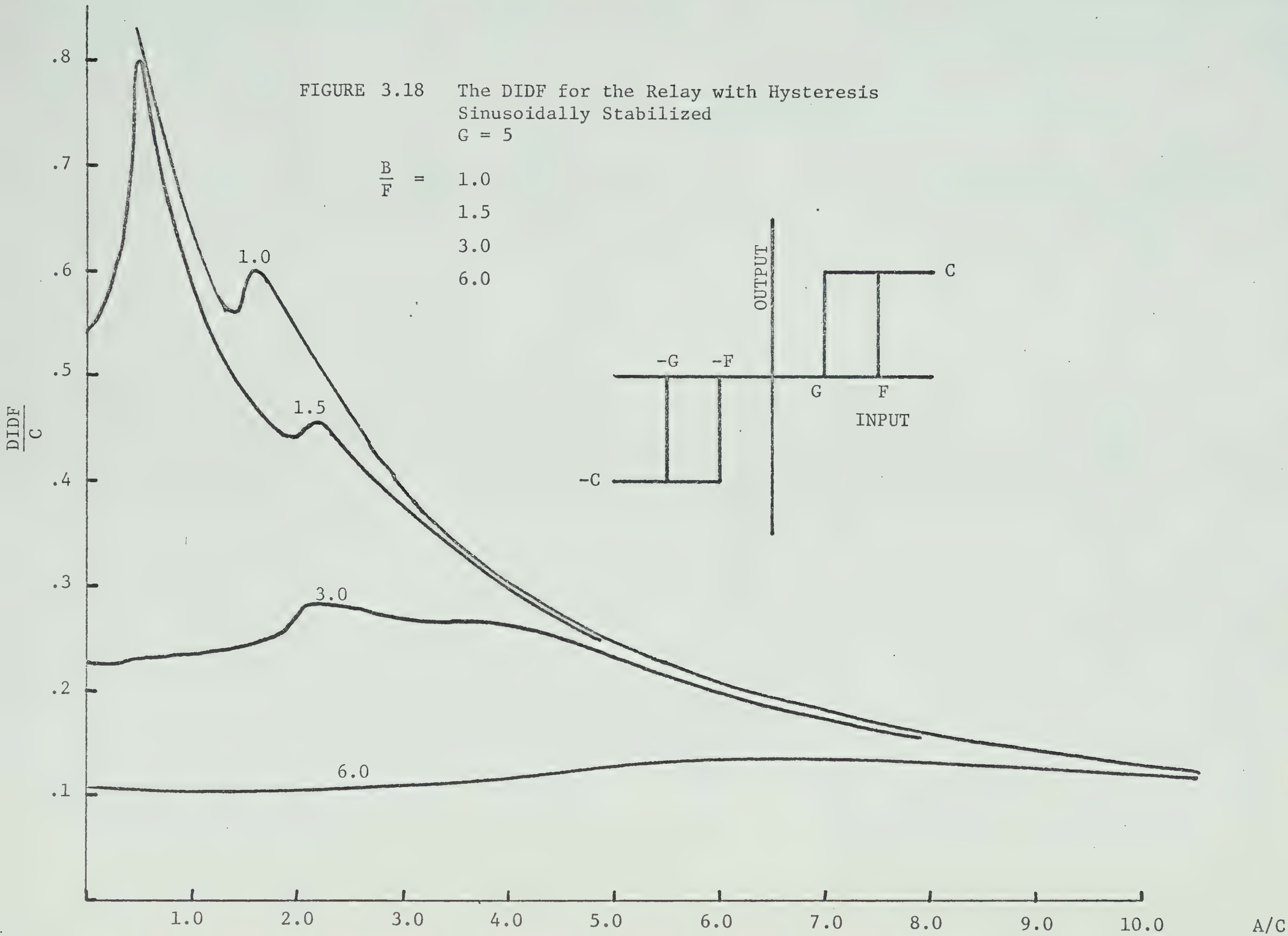
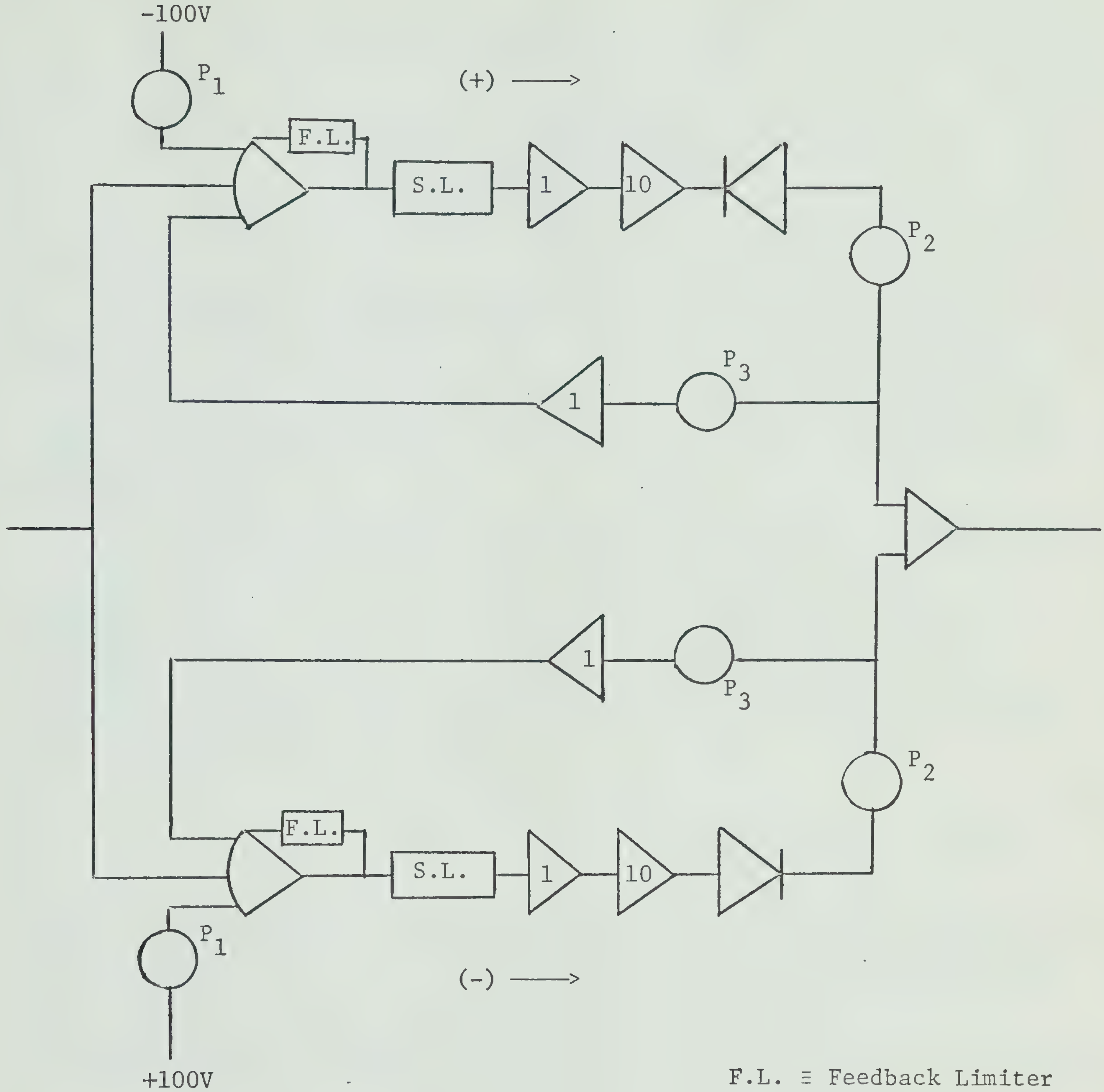


FIGURE 3.18 The DIDF for the Relay with Hysteresis Sinusoidally Stabilized
$G = 5$

$\frac{B}{F} =$ 1.0, 1.5, 3.0, 6.0

FIGURE 3.19 Analogue Representation of the Relay with Hysteresis



F.L. ≡ Feedback Limiter

S.L. ≡ Series Limiter

IV SOME STABILITY CONSIDERATIONS FOR THE RELAY WITH HYSTERESIS G = 5

In order to obtain an appreciation of the steps involved in applying the DIDF and simultaneously checking the applicability of the measurements for relay with hysteresis it is instructive to consider the following example.

Consider the relay with hysteresis with G = 5 in a simple feedback loop as shown in Fig. 1.2. Proceeding with an analysis in an analogous manner as for a single input describing function yields the following relationship describing the conditions under which the system will oscillate (Ref. 5, page 88).

$$G\ (iw) = \frac{-1}{\text{DIDF}} \qquad\qquad G\ (S) = \frac{K}{S(S+10)^2} \tag{4.1}$$

Now since the imaginary component of the DIDF was too small to measure one may obtain the frequency of oscillation by setting the imaginary part of $G\ (iw)$ equal to zero and solving for w.

$$G\ (iw) = \frac{K}{iw(iw+10)^2} = \frac{K}{-20w^2 + iw(100-w^2)}$$

$$= \frac{-20K}{400w^2 + (100-w^2)^2} - \frac{-i\ Kw(100-w^2)}{400w^4 + w^2(100-w^2)^2} \tag{4.2}$$

$$\text{Im } G(iw) = 0 = \frac{-i\,Kw(100 - w^2)}{400w^4 + w\,(100 - w^2)^2}$$

$$w = 10 \text{ rad/sec} \quad \text{or} \quad f = 1.59 \text{ HZ}$$

Additionally $\left|G(iw)\right| = \left.\frac{-K}{2000}\right|_{\omega = 10} = \frac{-1}{DIDF}$ (4.3)

$$DIDF = \frac{2000}{K} \tag{4.4}$$

Thus for a given $G(iw)$ with a known K one can find the DIDF (Eqn. 4.4). Knowing the DIDF it is a simple step to go to the appropriate graph which represents the DIDF for the particular stabilizing input being used and read $\frac{A}{C}$. The procedure is identical whether the stabilizing signal is sinusoidal or stochastic.

Some pertinent results have been tabulated in Table 1 for both sinusoidal and stochastic stabilizing signals. The value of K was chosen, the DIDF calculated from above and the amplitude of the auto-oscillation obtained from the DIDF graphs for various values of stabilizing signal. An attempt was made to obtain the value of K at which the auto-oscillation would extinguish itself. This value is of course easily attainable from the DIDF graphs and occurs either when A is zero, or the value of A at which the DIDF peaks and K is obtainable from equation 4.4. The amplitude of A was then checked experimentally on the analogue computer. For some DIDF's the exact extinguishing point was not attainable and so a range of possible values was given. Note the last extinguishing point for a stabilizing signal of $\frac{\sigma}{F} = 6.0$ was particularly difficult to obtain. No result was

TABLE I

STABILITY RESULTS FOR THE RELAY WITH HYSTERESIS G=5

i) Stochastically Stabilized

				Measured		
σ/F	K	DIDF	A/F	A/F	f	
1.0	10,000	.2	6.1	6.1	1.6	
1.0	4,000	.5	2.0	2.05	steady	
1.0	3,640-3,330	.55-.6	0	0		K = 3,400
1.5	10,000	.2	6.1	6.0		
1.5	5,700	.35	2.9	3.0		
1.5	4,350-4,100	.46-.49	0	0		K = 4,000
3.0	13,300	.15	7.6	7.5		
3.0	10,000	.20	5.0	5.0		
3.0	7,140-6,666	.28-.3	0	0		K = 6,800
6.0	20,000	.1	10.0	10.0		
6.0	14,300	.14	3.0	varied		
6.0	14,300	≈ .14	0	0	1.6	

TABLE I (continued)

ii) Sinusoidally Stabilized

				Measured		
B/F	K	DIDF	A/F	A/F	f	
1.0	5,000	.40	2.9	3.0	1.6	
1.0	2,660	.75	.7	7.5	steady	
1.0	1,000	2.0	0	0		K = 700
1.5	10,000	.20	6.2	6.5		
1.5	5,000	.40	2.7	2.9		
1.5	2,500	.80	0	0		K = 2,550
3.0	13,300	.15	8.4	8.5		
3.0	10,000	.20	5.9	6.0		
3.0	7,150	.28	0	0		K = 7,000
6.0	16,666	.12	9.5	9.5		
6.0	15,380	.13	8.4	8.5		
6.0	14,300	.14	0	0	1.6	K = 14,000

quoted because the DIDF is almost flat at low $\frac{A}{C}$ and hence $\frac{A}{C}$ can vary broadly without affecting the DIDF appreciably. The correspondence of the results was better than expected.

The graph for the DIDF for a dual sinusoid input into a relay with hysteresis exhibits definite peaks. These peaks have been investigated experimentally and it was found that if for a given DIDF two fundamental output amplitudes exist then the lower amplitude is unstable and will increase however slowly until the higher amplitude is reached. No attempt was made to measure the lower amplitude corresponding to a higher amplitude.

The instability of amplitudes where the DIDF has a positive slope is easily understood if one considers a small perturbation about a point on the graph. Thus if the graph has a positive slope and one moves upward one increment along the curve, then there is both an increase in the DIDF and also the amplitude. The net effect is regenerative and consequently the operating point will tend to shift. Similarly, if one moves downward one increment then both DIDF and amplitude output are decreased, consequently the net effect is to move away from the original operating point. A similar analysis of the DIDF curve with negative slope shows that a perturbation will tend to cancel itself since either the DIDF is increased and the amplitude decreased or vice versa. Thus ultimately a stable situation may exist.

V AN APPROXIMATE ANALYSIS OF A SPECIAL NONLINEAR SYSTEM

a) The Ideal Relay with Stochastic Stabilizing Signal

Both variable and fixed gain nonlinear systems exist in industry. Thus the output of the former may be controlled by varying the gain while the latter is controllable only by introducing an established signal to the nonlinearity. If the gain is varied the output cannot be easily predicted without using Sridhar's results which require the use of a digital computer, or the results which are given in this thesis. In any case there is no simple relationship between the two variables. If however the gain K of the system is adjusted so that the fundamental output of the nonlinearity is quite small and hence usually linear with respect to the fundamental input, depending on the value of the stabilizing signal then an approximation may be used.

As an example consider the ideal relay in the nonlinear system of Fig. 1.2 stabilized with a Gaussian noise input of standard deviation σ. The output fundamental amplitude will be given by Ref. 5.

$$P(A,\sigma) = \frac{2}{\sqrt{2\pi}} \; \frac{C}{\sigma} \; A_1F_1\left(\frac{1}{2};\ 2;\ \frac{-A^2}{2\sigma^2}\right) \tag{5.1}$$

where ${}_1F_1$ is the confluent hypergeometric series given by the expression

$$ {}_1F_1\left(K + \frac{1}{2};\ 2;\ \frac{-A^2}{2\sigma^2}\right) = \sum_{n=0}^{\infty} \frac{\Gamma(n + K + \frac{1}{2})\ \Gamma(2)}{(K + \frac{1}{2})\ \Gamma(n + 2)\ n!} \left(\frac{-A^2}{2\sigma^2}\right)^n $$

$$ {}_1F_1\left(\frac{1}{2};\ 2;\ \frac{-A^2}{2\sigma^2}\right) = 1 - \frac{1}{4}\ \frac{A^2}{2\sigma^2} + \frac{5}{80}\ \frac{A^4}{4 \cdot \sigma^4} - \ldots\ldots $$

Hence $$ \frac{P(A,\sigma)}{C\ A} = \frac{DIDF}{C} = \frac{2}{2\pi}\ \frac{1}{\sigma}\left[1 - \frac{1}{4}\ \frac{A^2}{2\sigma^2} + \frac{5}{320} \cdot \frac{A^4}{64} - \ldots\ldots\right] \qquad (5.2) $$

If $\frac{A}{\sigma}$ is chosen such that $\frac{A}{\sigma} \leq 2$ then if all terms beyond the second in the above series are neglected an error not larger than 6.25% will result in the DIDF/C.

Supposing a controllable noise signal existed with a minimum standard deviation σ_1, then the maximum value of A may be obtained which will make the approximation $\frac{A}{\sigma} \leq 2$ valid. Then a range of DIDF's exists over which the approximation is valid.

$$ \frac{3}{4} \cdot \frac{2}{\sqrt{2\pi}\ \sigma} \leq \frac{DIDF}{C} \leq \frac{2}{\sqrt{2\pi}\ \sigma} \qquad (5.3) $$

Thus a range of values of K for which permissible values of DIDF/C exist in the system may be obtained. Hence a K may be chosen for a particular system which will allow a simple analytical representation of the DIDF in terms of the two inputs.

In summary then, if there exists a nonlinear system such as in Fig. 1.2 a DIDF may be calculated simply from knowledge of the gain K, if K is a permissible value as established above. Hence the amplitude A of the system output may be calculated simply from the additional knowledge of σ the standard deviation of the stabilizing signal.

Example:

It is necessary to design a nonlinear system such as in Fig. 1.2 with an ideal relay which will have an output amplitude controllable by a stochastic noise signal which has a minimum value σ_1, RMS volts. The frequency of oscillation of the output may be calculated as on page 37. From equation 4.4 it follows that

$$\text{DIDF} = \frac{2000}{K} \tag{5.4}$$

Hence A maximum of the output is $\frac{A}{\sigma} \leqslant \sqrt{2}$ or A maximum is $A = \sigma_1 \sqrt{2}$ peak volts.

The range of values of permissible gain K may be obtained by substituting DIDF = 2000/K for this particular system in the $\frac{\text{DIDF}}{C}$ range giving

$$\frac{3}{4} \cdot \frac{2}{\sqrt{2\pi}\,\sigma_1} \leqslant \frac{2000}{CK} \leqslant \frac{2}{\sqrt{2\pi}\,\sigma_1}$$

$$\frac{1000\sqrt{2\pi}\,\sigma_1}{C} \leqslant K \leqslant \frac{4000\sqrt{2\pi}\,\sigma_1}{3C} \tag{5.5}$$

Supposing then a value K_1 in the indicated range is chosen then

$$\frac{\text{DIDF}_1}{C} = \frac{2000}{K_1 C}$$

But $\frac{\text{DIDF}}{C} = \frac{2}{\sqrt{2\pi}} \; \frac{1}{\sigma} \; [1 - \frac{1}{4} \; \frac{A^2}{2\sigma^2} \ldots$

Consequently A may be calculated within ± 6.25% in terms of σ to give

$$A = 2\sqrt{2}\,\sigma \sqrt{1 - \frac{\sqrt{2\pi}\,\sigma\, 2000}{2\, K_1 C}} \tag{5.6}$$

This value for A may seem to give a σ at which A will extinguish. This erroneous impression is caused by the approximation. Actually the extinguishing value of σ is given by

$$\frac{\sigma}{C} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\mathrm{DIDF}} \qquad \text{for a given value of K} \tag{5.7}$$

The above expression is easily obtained by setting $A = 0$ in the original expression for DIDF/C given by Eqn. 5.2.

b) The Ideal Relay with Sinusoidal Stabilizing Signal

The DIDF curves for the ideal relay with a sinusoidal stabilizing signal exhibit definite peaks. Operating amplitudes for a simple feedback system as in Fig. 1.2 smaller than the amplitude at which the DIDF peaks are not attainable because the network is unstable. The DIDF curve has a positive slope which causes the amplitude of the output to increase until the slope turns negative.

If the feedback of the network is changed to positive, to attempt to compensate for the positive slope, the system output increases or decreases without bound. The operating points on the DIDF vs. A/C curve which has a positive slope are not realizable with positive or negative feedback.

Nevertheless, to check experimental results it is interesting to calculate the DIDF when A/C = 0. The normalized DIDF with positive slope is given by Ref. 5 as:

$$\frac{DIDF}{C} = \frac{8}{\pi^2} \frac{B}{A^2} \left[E\left(\frac{A}{B}\right) - \left(1 - \frac{A^2}{B^2}\right) K\left(\frac{A}{B}\right)\right] \quad B \leq A \qquad (5.8)$$

where $E\left(\frac{A}{B}\right)$ is the complete elliptical integral of the second kind

$K\left(\frac{A}{B}\right)$ is the complete elliptical integral of the first kind.

Set $A = 0$ and by l'Hospital's rule differentiate the numerator and the denominator twice with regard to A to obtain the limiting value of DIDF/C when A goes to zero.

The limiting value becomes

$$\frac{DIDF}{C} = \frac{2}{\pi B} \qquad (5.9)$$

This value of normalized DIDF when A/C = 0 agress for practical purposes with the measured value.

If the system has a negative feedback then self oscillation will extinguish at the peak value of the DIDF curve.

A curve representing the peak values of the DIDF vs. the amplitude output for the ideal relay with sinusoidal stabilizing signal may be calculated as follows. The part of the curve at which the DIDF peaks is given by the following equation:

$$\frac{DIDF}{C} = \frac{8}{\pi^2} \frac{1}{A} E(B/A) \qquad B \leq A \quad \text{(Ref. 5)} \qquad (5.10)$$

Differentiate the equation with respect to A and set it equal to zero to solve for the amplitude at which the DIDF/C has a peak value.

$$\frac{\partial}{\partial A} \frac{DIDF}{C} = - \frac{8}{\pi^2} \frac{E(B/A)}{A^2} + \frac{8}{\pi^2} \frac{1}{A} \frac{dE(B/A)}{d\ B/A} \frac{\partial(B/A)}{\partial A}$$

$$0 = E(B/A) + \frac{B}{A} \frac{dE(B/A)}{B/A} \qquad (5.11)$$

This equation was solved on the digital computer available at the University of Alberta to give

$$B/A = 0.9089$$

The peak values of the DIDF for a given stabilizing signal then occur at a fixed amplitude ratio B/A satisfying equation 5.11. The equation of the curve passing through the peaks of the DIDF vs. A/C curve now becomes

$$DIDF = \frac{8}{\pi^2} \frac{1.0974}{A}$$

VI AN APPLICATION OF THE DIDF TO THE ORDINARY DESCRIBING FUNCTION

The use of the ordinary describing function for a nonlinear autonomous system depends upon the ideality of the low pass filter. To check the ideality of any low pass filter and hence the correctness of the describing function the DIDF may be used. The second harmonic which passes the low pass filter may be used as the stabilizing signal in the DIDF. The simplest low pass filter to examine would of course be the integrator $G(S) = 1/S$. However, oscillation can only occur for a double valued nonlinearity where the equation $G(S) = -\frac{1}{Keq}$ (where Keq is the common describing function) can be satisfied. Since, in practice, the imaginary component output of a nonlinearity is usually quite small and variable over a small range then the frequency of oscillation of the autonomous system would vary. The imaginary component of a nonlinearity is usually variable since the practical relay sticks to give it hysteresis. The analogue model on the other hand has a finite rise time and settling time which affect the imaginary component sufficiently to make calculated results impractical.

A more practical plant $G(S) = \frac{1}{S(S+1)^2}$ may be chosen in conjunction with an ideal relay in the configuration of Fig. 1.2 to test the common describing function. The frequency of oscillation may easily be obtained by setting the imaginary component of $G(iw)$ equal to zero hence

w = 1.

Then $|G\ (i1)| = \frac{1}{1 \times 2} = \frac{1}{2}$

Then $|G\ (i3)| = \frac{\alpha 1}{3 \times 10} = \frac{\alpha}{30}$ since the lowest harmonic has a frequency $w = 3$ rad./sec. Then the amplitude ratio of the two outputs will be

$$\frac{|G\ (i3)|}{|G\ (i1)|} = \frac{2}{90} = \frac{1}{45}$$

where α is the amplitude ratio of the second harmonic to the first. Now the fundamental output of the nonlinearity is given by Ref. 5 as

$$P(A,B) = \frac{8}{\pi^2} C\ E(B/A)$$

where A is the fundamental amplitude

B is the stabilizing amplitude

P is the fundamental output of the nonlinearity

E(B/A) is the complete elliptical integral of the second kind

Now $B/A = \frac{1}{45}$ and the complete elliptical integral of the second kind may be approximated by

$$E(B/A) = \frac{\pi}{2} [1 - (\tfrac{1}{2})^2 (B/A)^2 \ .....$$

Then

$$P(A,B) = \frac{8}{\pi^2} C \frac{\pi}{2} [1 - \frac{1}{4} (\tfrac{1}{45})^2 - \ .....$$

Consequently there is less than 0.1% error in the amplitude as predicted by the ordinary describing function for this particular example.

CONCLUSION

A simple convenient method for the measurement of dual input describing functions was given for any time and frequency independent nonlinearity.

The method was applied to five different nonlinearities; the relay, the relay with deadband, the limiter, the limiter with deadband and a relay with hysteresis for both a dual sinusoid input and a sine wave plus Gaussian noise combination. The results were displayed on graphs. The curves of normalized DIDF vs. normalized fundamental amplitude for the relay, the relay with deadband, the limiter and the limiter with deadband for a dual sine wave input compared favourably with Boyer's results (Ref. 4) which were obtained by an approximate analytical method. The curves of normalized DIDF vs. normalized fundamental amplitude for the relay, the relay with deadband, the limiter, and the limiter with deadband for a sine wave input plus a Gaussian noise input compared favourably with the analytical expansions obtained by Sridhar (Ref. 5).

A relay with hysteresis was simulated on the analogue computer which approached the ideal kind in accuracy. However, the error for low input amplitudes was greater than 10% and consequently the relay was termed unique and the DIDF vs. the input amplitude graphed for both a dual sinusoid input and a sinusoid plus Gaussian noise input. These unique results were tested by comparing experimental results for amplitude and frequency of

self oscillation in a feedback loop as in Fig. 1.2 to the results obtained from the graphs. The comparison as given in Table 1 showed no evident discrepancies.

The imaginary component of the DIDF for the relay with hysteresis was not measurable agreeing with the results stated by Boyer that the phase shift of the relay with hysteresis is zero (Ref. 4).

While measuring the amplitude of the fundamental output of the nonlinearities mentioned it was noted that for the stochastic plus sinusoid input the fundamental output varied very nearly linearly with respect to the fundamental input over a given range which could be extended by increasing the stochastic signal. This result suggested the approximate analysis which was undertaken for the ideal relay. The same type of analysis may be undertaken for the relay with deadband, the limiter and the limiter with deadband for which analytical results exist.

In addition the DIDF of the relay, relay with deadband, limiter and limiter with deadband can be monotonic with respect to the fundamental input for a dual input of sine wave plus Gaussian noise. If then the nonlinearity is used in a feedback loop such as in Fig. 1.2 and the Gaussian noise is used as the stabilizing signal, then the auto oscillation amplitude can be controlled over the complete positive range of values. For nonlinearities with a dual sine wave input the auto oscillation cannot be similarly controlled. The amplitude may exhibit jump phenomena or extinguish at a finite value due to the peaks in the DIDF vs. fundamental input amplitude curves.

The dual input describing function was also applied to an example to obtain an appreciation of the accuracy of the ordinary describing function.

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